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RESEARCH ARTICLE

AN EFFICIENT ALGORITHM FOR FINDING MIXED NASH EQUILIBRIA IN 2-PLAYER GAMES

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ABSTRACT

This paper presents an algorithm for calculating mixed Nash equilibria in 2-player games. The algorithm is based on the mathematical equivalence between the expected payoff function of bi-matrix games and the fuzzy average. It was proved that the expected payoff function of 2-player games is identical to the fuzzy average of two linguistic values when the payoff matrix is replaced with the consequence matrix, the strategy sets are replaced with term sets in linguistic variables. This paper proves that the new algorithm can compute mixed NE in 2-player games within polynomial time for any types of bi-matrix games. We claim that there is a fully polynomial time scheme for computing mixed NE in 2-player games.

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INTRODUCTION

Recent sequence of papers shows that computing one NE is PPAD (Polynomial Parity Arguments on Directed graphs)-complete for two, three, or four player games in strategic form [7] [8] [14] [15]. Chen and Deng [8] show that computing NE for two player games is PPAD-complete. Daskalakis et al. [14] show that finding NE for four player games is PPAD complete. Papers [7] and [15] independently show that calculating NE for three player games is PPAD-complete. All known algorithms require exponential time in the worst case. Chen et al. [9] show that the problem of computing a (1/n)-well-supported NE in polymatrix game is PPAD-complete. Daskalakis and Papadimitriou presented a polynomial time approximation scheme (PTAS) for ε-approximation NE in anonymous games. In [16] the authors described a PTAS for finding an ε-approximation NE in an anonymous game with two pure strategies with a certain order of running time. The PTAS in [16] depends on the existence of an ε-approximation NE consisting of integer multiples of ε^2. Daskalakis [13] presented improved PTAS from the running time view of point. This improved PTAS is based on the existence of an ε-approximation NE satisfying the following conditions: either at most $\alpha(1/\varepsilon^3)$ players play mixed strategies, or al players who mix play the same mixed strategy. In [17] the authors extended the PTAS with any bounded number of pure strategies with running time $n^{g(\alpha,1/\varepsilon)} \bullet U$ for some function g of α, number of pure strategies and $1/\varepsilon$, where U denotes the number of bits required to describe the payoff. Daskalakis and Papadimitriou's PTAS [13] [16] [17] are algorithms that enumerate a set of mixed strategy profiles which is independent of the input game as candidates for approximation NE, that is, the game is used only to verify if a given mixed strategy profile is an ε-approximation NE. The PTAS is called oblivious algorithms [18]. Daskalakis and Papadimitriou showed that these type of PTAS for anonymous games must have running time exponential in $1/\varepsilon$ [18]. They also proposed a non-oblivious PTAS for two-strategy anonymous games.

Chen et al. [10] presented that the problem of computing a $1/\varepsilon$-will-supported NE in a polymatrix game is PPAD-complete. Chen et al. [9] showed that the problem of finding an ε-approximation NE in an anonymous game with seven pure strategies is PPAD-complete.

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The application of fuzzy theory to decision making problems initiated by Bellman and Zadeh [2] in 1970. Butnariu [5] did fundamental research on fuzzy games. Chakeri and Sheikholeslam [6] show a method of finding fuzzy NE in Crisp and fuzzy games, Garagic et al. [23] extend the concept of non cooperative game theory to fuzzy non cooperative games under uncertainty phenomena. Wu and Soo [35] applied fuzzy game theory to multi-agent coordination.

In this article, we use fuzzy theory as a tool to apply the fuzzy average to 2-player games. This paper represents a revised algorithm for computing mixed NEs in 2-player games [20] [21]. This algorithm is based on the relationship between the expected payoff function of 2-player games and the mathematical representation of the fuzzy average of two linguistic values. Based on author's understanding, the problem of computing mixed NE in 2-player games in normal form has not been proved in P-complete class. We prove that the new algorithm can compute mixed NEs in 2-player games in polynomial time for any types of 2-player games in normal form. We claim that there is a fully polynomial time scheme for 2-player games in normal form.

This article is organized as follows. Section 2 discusses the preliminaries. Section 3 describes the algorithm for calculating mixed NEs in 2-player games in details. Associated with the algorithm, a theorem, which indicates the main result in this article, and its proof are represented in this section. Section 4 provides examples. Section 5 is the conclusion and future study.

PRELIMINARIES

2-player games in normal form

2-player games in normal form are also called bi-matrix games. A 2-player game is denoted by $G=(2,\ \{S_i\}_{i\in 2},\ \{u_i\}_{i\in 2})$, where $S_1=(s_{11},s_{12},...,s_{1k})$, $S_2=(s_{21},s_{22},...,s_{2l})$ is a set of strategies for player 1, player 2 respectively; the expected payoff function u_1 of player 1, u_2 of player 2 is as follows.

$$\begin{cases} u_1(P_1,P_2)=P_1 \bullet A_{12} \bullet P_2^T \\ u_2(P_2,P_1)=P_2 \bullet A_{21} \bullet P_1^T \end{cases} \qquad (2.1)$$

where $P_1=\{p_{11},...,p_{1k}\}\in \Delta_k(P_1):=\{(p_{11},p_{12},...,p_{1k}) \mid \sum_{i=1}^{k} p_{1i}=1; p_{1i}\geq 0 (i=1,2,...,k)\}$ represents the probability distribution over S_1; $P_2=\{p_{21},...,p_{2l}\}\in \Delta_l(P_2)$ represents the probability distribution over S_2; A_{12}, A_{21} is $k\times l$ payoff matrix, $l\times k$ payoff matrix, of player 1, player 2, respectively.

Optimal values of function *f(x, y)*

Let us review the Taylor expansion of a two variable function. Suppose that function *f(x, y)* is an infinitely differentiable, and *(a, b)* is a critical point of *f(x, y)*, with $f_x(a,b)=f_x(a,b)=0$. Function *f(x, y)*'s Taylor expansion is as follows.

$$f(x,y)=f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)$$

$$+\frac{1}{2}f_{xx}(a,b)(x-a)^2+\frac{1}{2}f_{yy}(a,b)(y-b)^2+f_{xy}(a,b)(x-a)(y-b)+H.O.T \text{ (high order terms for short)}$$

In the case that *(a, b)* is a critical point, the first derivatives $f_x(a,b)$ and $f_x(a,b)$ are zero, the above equation becomes

$$f(x,y)-f(a,b)=\frac{1}{2}A(x-a)^2+\frac{1}{2}C(y-b)^2+B(x-a)(y-b)+H.O.T$$

where $A=f_{xx}$, $B=f_{yy}$ and $C=f_{xy}$. We can ignore the high order terms when *(x, y)* sufficiently close to *(a, b)*. We rewrite the result as

$$f(x,y)-f(a,b)=\frac{1}{2}(x-a \quad y-b)\bullet\begin{pmatrix} A & B \\ B & C \end{pmatrix}\bullet\begin{pmatrix} x-a \\ y-b \end{pmatrix}=\frac{1}{2}h^T Hh \qquad (2.2)$$

where $h^T = (x-a, \; y-b)$, the matrix H is called Hessian matrix, the representation on right hand is called quadratic form. $D = Det\,H = \begin{vmatrix} A & B \\ B & C \end{vmatrix} = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2$ is the determinant of Hessian matrix.

There are three cases we need to consider.

1) If *(a, b)* is local minimum value, then the right hand side of (2.2) must be positive for all *(x, y)* in a neighborhood of *(a, b)*, such as $H > 0$ for all *(x, y)* in a neighborhood of *(a, b)*. Based on linear algebra, if the two eigenvalues of matrix *H* are positive, then $H > 0$.
2) If *(a, b)* is local maximum, then the right hand side of (2.2) must be negative, such as such as $H < 0$ for all *(x, y)* in a neighborhood of *(a, b)*. Based on linear algebra, if the two eigenvalues of matrix H are negative, then $H < 0$.
3) If *(a, b)* is a saddle point, then the right hand side of (2.2) is either positive or negative depending on the values in neighborhood of *(a, b)*. According to linear algebra, when two eigenvalues of matrix *H* are nonzero and have opposite sigh, point *(a, b)* is a saddle point.

The above inferences can be rephrased for two variable functions as follows.

1) If D > 0 and A > 0, then *f(a, b)* is a local minimum of *f(x, y)*;
2) If D > 0 and A < 0, then *f(a, b)* is a local maximum of *f(x, y)*;
3) If D < 0, then (a, b) is a saddle point of *f(x, y)*;
4) If D = 0, then no conclusion can be drawn.

1.1 Fuzzy Numbers

A fuzzy number is a fuzzy set which is defined in *R*. There are some types of fuzzy numbers [27]. For example, triangular fuzzy numbers (TFNs), trapezoid fuzzy numbers, and etc. We only review TFNs in this paper.

The definition of a TFN:

A TFN is denoted as *(a, b, c)*, where $a \in R; b \in R; c \in R$ and $(a \le b \le c)$. The TFN *(a, b, c)* is described in Figure 1.

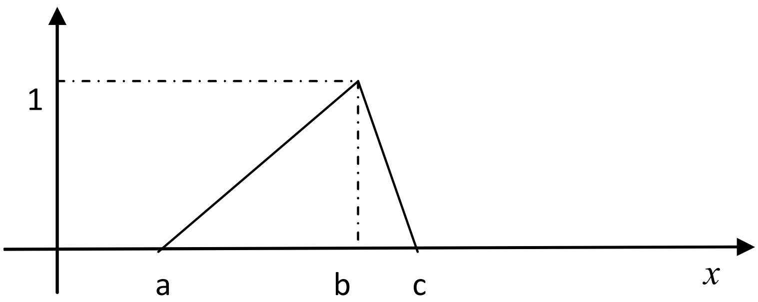

Figure 1, Triangular fuzzy number *(a, b, c)*

The membership function of the TFN *(a, b, c)* is defined as $\mu(x) = \begin{cases} \dfrac{x-a}{b-a} & if \quad x \in [a,b) \\ \dfrac{c-x}{c-b} & if \quad x \in (b,c] \\ 0 & otherwise \end{cases}$

There are two special TFNs. One is *(a, a, c)*, such as *a = b,* and the other is *(a, c, c)*, such as *b = c.* Suppose the membership function of TFN *(a, a, c)*, *(a, c, c)* is $\mu(x)$, $v(x)$, respectively. Then one has the following.

$\mu(x) = -\frac{1}{c-a}x + \frac{c}{c-a}$; *(a <c),* $\nu(x) = \frac{1}{c-a}x - \frac{a}{c-a}$; *(a <c).*

The Fuzzy Average

The fuzzy average [22] is defined with the average of linguistic values of a linguistic variable *(x, T(x), U, G, M)*[36] [37], where x the name of the variable; *T(x)* is the term set of *x*, *U* is the universe of discourse which is usually defined as interval [0, 1]; *G* is the syntactic rule which generates the terms in *T(x)*; *M* is a semantic rule which is usually a mapping from the set *T(x)* to a set of fuzzy numbers defined in *U*. The fuzzy average of two values of a linguistic variable was described as follows.

Suppose that the two values of a linguistic variable are as follows. $(x, T_1(x), U_1, G_1, M_1)$ and $(y, T_2(y), U_2, G_2, M_2)$. $M_1 : T_1(x) \to \{A_1, A_2, \ldots, A_n\}, M_2 : T_1(y) \to \{B_1, B_2, \ldots, B_m\}$, where A_i *(i=1,…,n)* and B_j *(j=1, 2,…,m)* are triangular fuzzy numbers (TFNs) which are defined on $U_1 = [0, 1]$ and $U_2 = [0, 1]$, $\mu_{A_i}(x)$ and $\mu_{B_j}(y)$ are the membership functions of TFNs A_i and B_j, respectively. The fuzzy average is defined as:

$$u(x,y) = \sum_{i=1}^{n}\sum_{j=1}^{m} \mu_{A_i}(x)\mu_{B_j} r_{ij} = \mu_A(x) \bullet R \bullet \mu_B(y)$$

where

$$\mu_A(x) = \{\mu_{A_1}(x_1), \mu_{A_2}(x_1), .. \mu_{A_n}(x_1)\}, \mu_A(x) \in \Delta_n(\mu_A(x));$$
$$\mu_B(y) = \{\mu_{B_1}(y_1), \mu_{B_2}(y_1), .. \mu_{B_m}(y_1)\}, \mu_B(y) \in \Delta_m(\mu_B(y)).$$

$x \in U_1$ and $y \in U_2$; *n* is the number of entries in $T_1(x)$; *m* is the number of entries in $T_2(y)$; $R = (r_{ij})$ is a $n \times m$ matrix, which is called the consequence matrix [22]. It was proved that the fuzzy average converges to arithmetic mean under specific conditions [22].

$\mu_{A_i}(x)$ is interpreted as the weight of element $x_i \in T_1(x)$. For a given $x \in U_1$ and $y \in U_2$, the vector $\mu_A(x)$, $\mu_B(y)$ is interpreted as probability distribution over $T_1(x)$, $T_2(y)$, respectively.

In game theory, the set of strategies $S_i = (s_{i1}, \ldots s_{ik})(i = 1,2,..n)$ can be interpreted as the term set $T(S_i)$ in the concept of linguistic variables. For example, for a rock-scissors-paper game, a player's strategy set $S = (r, s, p)$ can be considered as term set of {rock, scissors, paper} in linguistic variables, such as $T_1(S) = \{r, s, p\}$.

For two player games in normal form, when player's strategy set is represented by the term set, and the payoff matrix is same as the consequence matrix, it was proven that the expected payoff function is identical to the fuzzy average [20].

The algorithm of computing Nash Equilibria in 2-player games

This new algorithm is an extension of the algorithm [20] [21] for 2-player games. The main idea is the relationship between the expected payoff function of 2-player games in strategic form and the concept of the fuzzy average. Paper [20] proved that the expected payoff function of 2-player games in strategic form is identical to the fuzzy average of two linguistic values when the strategy sets in 2-player games are represented with the term sets in linguistic variables, the payoff matrix is replaced with the consequence matrix, and the probability distribution over strategy set for each player is represented with the semantic rule *M* in linguistic variables. The algorithm in this paper improves the algorithm which was published in [20] from the point of simplicity. The requirement of dividing strategy domains is no longer required for 2-player games in normal form. The algorithm is as follows.

1. For a given 2-player games in normal form, build two linguistic values $(S, T(S_i), U_i, G_i, M_i)(i = 1,2)$.
1) Define the term sets by using the strategy sets, such as $T(S_i) = \{S_i\}, (i = 1,2)$.

2) Define $U_i = [0,\ 1](i = 1,2)$, and suitable TFNs $E_i\,(i = 1,2,...,\ k)$ and $H_i\,(i = 1,2,...,\ l)$, and their membership functions $\mu_i(x)(i = 1,2,...k)\ x \in U_1$ and $\nu_i(y)(i = 1,2,...,\ l)\ y \in U_2$, respectively.

3) Define proper semantic rules $M_i : T(S_i) \rightarrow P_i(i = 1,2)$ where P_i is defined as

$$P_1(x) = \left(\frac{\mu_1(x)}{\sum_{j=1}^{k}\mu_j(x)}, \frac{\mu_2(x)}{\sum_{j=1}^{k}\mu_j(x)}, ..., \frac{\mu_k(x)}{\sum_{j=1}^{k}\mu_j(x)}\right) \qquad (3.1)$$

$$P_2(y) = \left(\frac{\nu_1(y)}{\sum_{j=1}^{k}\nu_j(y)}, \frac{\nu_2(x)}{\sum_{j=1}^{k}\nu_j(y)}, ..., \frac{\nu_l(y)}{\sum_{j=1}^{l}\nu_j(y)}\right) \qquad (3.2)$$

It is clear that $P_1 \in \Delta k$ and $P_2 \in \Delta_l$.

2. Construct the expected payoff function (2.1) by using the given payoff matrices, and the probability distributions which are defined in (3.1) and (3.2).
3. Solve (3.3).

$$\begin{cases} \dfrac{\partial u_1(P_1, P_2)}{\partial P_1} = 0 \\ \dfrac{\partial u_2(P_2, P_1)}{\partial P_2} = 0 \end{cases} \qquad (3.3)$$

3. Verify that the solution (P_1^*, P_2^*) of (3.3) satisfies the conditions: $P_i^* \in \Delta; (i = 1,2)$ or $x \in U_1; y \in U_2$

4. Examine the point (P_1^*, P_2^*) by using Hessian matrix as follows.

5.1. For 2-player games, one can calculate the following.

$$A_1 = \frac{\partial^2 u_1(P_1, P_2)}{\partial P_1^2}\bigg|_{\substack{P_1 = P_1^* \\ P_2 = P_2^*}}, B_1 = \frac{\partial^2 u_1(P_1, P_2)}{\partial P_1 \partial P_2}\bigg|_{\substack{P_1 = P_1^* \\ P_2 = P_2^*}}, \text{ and } C_1 = \frac{\partial^2 u_1(P_1, P_2)}{\partial P_2^2}\bigg|_{\substack{P_1 = P_1^* \\ P_2 = P_2^*}}$$

$$A_2 = \frac{\partial^2 u_2(P_2, P_1)}{\partial P_2^2}\bigg|_{\substack{P_1 = P_1^* \\ P_2 = P_2^*}}, B_2 = \frac{\partial^2 u_2(P_2, P_1)}{\partial P_2 \partial P_1}\bigg|_{\substack{P_1 = P_1^* \\ P_2 = P_2^*}}, \text{ and } C_2 = \frac{\partial^2 u_2(P_2, P_1)}{\partial P_1^2}\bigg|_{\substack{P_1 = P_1^* \\ P_2 = P_2^*}}$$

5.2. Calculate $D_i = A_i C_i - B_i^2 (i = 1,2)$.

If $D_i < 0$, then (P_1^*, P_2^*) is a saddle point of $u_i(P_i, P_{-i})$;
If $D_i > 0$ and $A_i < 0$, then, $u_i(P_i, P_{-i})$ reaches a local maximum value at (P_1^*, P_2^*);
If $D_i > 0$ and $A_i > 0$, then $u_i(P_i, P_{-i})$ reaches a local minimum value at (P_1^*, P_2^*);
If $D_i = 0$, then no decision can be made.

This algorithm is able to calculate mixed NEs in 2-player games within polynomial time. We give the following theorem.

Theorem 1

For a given 2-player game in normal form, when probability distributions of player 1 and player 2 are defined with (3.1) and (3.2), then the algorithm can find mixed NEs in the 2-player game in polynomial time.
If we can prove that (3.3) becomes a system of linear equations, then the theorem is proved [3].

Proof:

when $P_1(x)$ and $P_2(y)$ are defined with (3.1) and (3.2), then (3.3) becomes the following.

$$\begin{cases} \dfrac{\partial u_1(P_1,P_2)}{\partial P_1} = \dfrac{dP_1(x)}{dx} \bullet A_{12} \bullet P_2(y)^T = 0 \\ \dfrac{\partial u_2(P_2,P_1)}{\partial P_2} = \dfrac{dP_2(y)}{dy} \bullet A_{21} \bullet P_1(x)^T = 0 \end{cases} \qquad (3.4)$$

According to the property of TFNs, we can define $\mu_j(x),(j=1,2,\ldots,k,x\in U_1)$ and $\nu_j(y),(j=1,2,\ldots,l,y\in U_2)$ as follows.

$\mu_j(x)=a_jx+b_j, \nu_j(y)=c_jy+d_j$, where $a_j \in R$, $b_j \in R$, $c_j \in R$ and $d_j \in R$ are constant.

Then the sum of $\mu_j(x)$, $\nu_j(y)$ is calculated respectively.

$$\sum_{l=1}^{k} \mu_l(x) = (\sum_{l=1}^{k} a_l)x + \sum_{l=1}^{k} b_l,\ \sum_{l=1}^{k} \nu_l(y) = (\sum_{l=1}^{k} c_l)y + \sum_{l=1}^{k} d_l$$

(3.1) and (3.2) become as follows.

$$P_1(x) = (\frac{\mu_1(x)}{\sum_{l=1}^{k}\mu_l(x)}, \frac{\mu_2(x)}{\sum_{l=1}^{k}\mu_l(x)}, \ldots, \frac{\mu_k(x)}{\sum_{l=1}^{k}\mu_l(x)}) = (\frac{a_1x+b_1}{(\sum_{l=1}^{k}a_l)x+\sum_{l=1}^{k}b_l}, \frac{a_2x+b_2}{(\sum_{l=1}^{k}a_l)x+\sum_{l=1}^{k}b_l}, \ldots, \frac{a_kx+b_k}{(\sum_{l=1}^{k}a_l)x+\sum_{l=1}^{k}b_l}).$$

$$P_2(y) = (\frac{\nu_1(y)}{\sum_{l=1}^{k}\nu_l(y)}, \frac{\nu_2(y)}{\sum_{l=1}^{k}\nu_l(y)}, \ldots, \frac{\nu_l(y)}{\sum_{l=1}^{k}\nu_l(y)})$$

$$= (\frac{c_1y+d_1}{(\sum_{l=1}^{k}c_l)y+\sum_{l=1}^{k}d_l}, \frac{c_2y+d_2}{(\sum_{l=1}^{k}c_l)y+\sum_{l=1}^{k}d_l}, \ldots, \frac{c_ky+d_k}{(\sum_{l=1}^{k}c_l)y+\sum_{l=1}^{k}d_l}).$$

Then the derivative of $P_1(x)$, $P_2(y)$ is the following.

$$\frac{dP_1(x)}{dx} = (\frac{a_1\sum_{l=1}^{k}b_l - b_1\sum_{l=1}^{k}a_l}{((\sum_{l=1}^{k}a_l)x+\sum_{l=1}^{k}b_l)^2}, \frac{a_2\sum_{l=1}^{k}b_l - b_2\sum_{l=1}^{k}a_l}{((\sum_{l=1}^{k}a_l)x+\sum_{l=1}^{k}b_l)^2}, \ldots, \frac{a_k\sum_{l=1}^{k}b_l - b_k\sum_{l=1}^{k}a_l}{((\sum_{l=1}^{k}a_l)x+\sum_{l=1}^{k}b_l)^2})$$

$$\frac{dP_2(y)}{dy} = (\frac{c_1\sum_{l=1}^{k}d_l - d_1\sum_{l=1}^{k}c_l}{((\sum_{l=1}^{k}c_l)y+\sum_{l=1}^{k}d_l)^2}, \frac{c_2\sum_{l=1}^{k}d_l - d_2\sum_{l=1}^{k}c_l}{((\sum_{l=1}^{k}c_l)y+\sum_{l=1}^{k}d_l)^2}, \ldots, \frac{c_k\sum_{l=1}^{k}d_l - d_k\sum_{l=1}^{k}c_l}{((\sum_{l=1}^{k}c_l)y+\sum_{l=1}^{k}d_l)^2})$$

It is clear that the elements in each derivative vector have common denominator, the numerator of each element in the above derivative vectors is a constant. On the other hand, the elements in vector $P_i(i=1,2)$ have common denominator, and the numerator of each element in vector $P_i(i=1,2)$ is a piecewise linear function. Because we can ignore the common denominators in each equation of (3.4), then the first equation in (3.4) finally becomes a linear equation of *y*, the second equation in (3.4) becomes a linear equation of *x*. Therefore, equation (3.4) becomes a system of linear equations of *x* and *y*. Q.E.D.

EXAMPLES

Three examples are described in this section.

Example 1 - Rock- Scissors-Paper game. Find mixed NEs for a Rock-Scissors-Paper Game with the following payoff bi-matrix.

$$\begin{pmatrix} & rock & paper & scissors \\ rock & (0,0) & (1,-1) & (-1,1) \\ paper & (-1,1) & (0,0) & (1,-1) \\ scissors & (1,-1) & (-1,1) & (0,0) \end{pmatrix}$$

This is a 2-player symmetric game. The payoff matrices of player 1 and player 2 are as follows.

$$A_{12} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}, \; A_{21} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}^T = A_{12}.$$

We build two linguistic values $(S_i, T_i(S_i), U_i, G_i, M_i)(i=1,2)$. $T_i(S_i) = (rock, scissors, paper)$, $U_i = [0,\ 1](i=1,2)$, and semantic rule M_i is defined as follows.

$M_1 : T_1(S_1) = (rock, scissors, paper) \rightarrow (E_1, E_2, E_3)$
$M_2 : T_2(S_2) = (rock, scissors, paper) \rightarrow (H_1, H_2, H_3)$, where E_i, $H_i (i = 1,2,3)$ is a TFN defined in U_1, U_2, respectively.

We define $E_1 = (0,0,1)$, $E_2 = E_3 = (0,1,1)$, and $H_1 = H_2 = (0,0,1)$, $H_3 = (0,1,1)$. Then, the sum of membership functions $\mu_i(x)$, $\nu_i(y)$ of E_i, H_i is as follows.

$$\sum_{l=1}^{3} \mu_l(x) = x+1, \; \sum_{l=1}^{3} \nu_l(y) = 2-y.$$

$$P_1(x) = (\frac{1-x}{1+x}, \frac{x}{1+x}, \frac{x}{1+x}), \text{ and } \frac{dP_1(x)}{dx} = (\frac{-2}{(1+x)^2}, \frac{1}{(1+x)^2}, \frac{1}{(1+x)^2}),$$

$$P_2(y) = (\frac{1-y}{2-y}, \frac{1-y}{2-y}, \frac{y}{2-y}), \text{ and } \frac{dP_2(y)}{dy} = (\frac{-1}{(2-y)^2}, \frac{-1}{(2-y)^2}, \frac{2}{(2-y)^2}).$$

One can solve the following system of linear equations.

$$\begin{cases} \dfrac{\partial u_1(P_1, P_{-1})}{\partial P_1} = \dfrac{dP_1(x)}{dx} \bullet A_{12} \bullet P_2(y)^T = K_1(-2,\ 1,\ 1) \bullet \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \bullet \begin{pmatrix} 1-y \\ 1-y \\ y \end{pmatrix} = 0 \\ \dfrac{\partial u_2(P_2, P_{-2})}{\partial P_2} = \dfrac{dP_2(y)}{dy} \bullet A_{21} \bullet P_1(x)^T = K_2(-1,\ -1,\ 2) \bullet \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \bullet \begin{pmatrix} 1-x \\ x \\ x \end{pmatrix} = 0 \end{cases}$$

where $K_1 = \dfrac{1}{(1+x)^2(2-y)}$ and $K_2 = \dfrac{1}{(2-y)^2(1+x)}$.

The solution is $x=\frac{1}{2}\in[0,1]$, $y=\frac{1}{2}\in[0,1]$. The mixed NE (P_1^*, P_2^*) with probability distributions $P_1^* = (\frac{1}{3},\frac{1}{3},\frac{1}{3})$ and $P_2^* = (\frac{1}{3},\frac{1}{3},\frac{1}{3})$.

Let us examine the solution (P_1^*, P_2^*) for player 1 by using Hessian matrix or step 5 described in the algorithm.

$$A_1 = \frac{\partial^2 u_1(P_1,P_2)}{\partial P_1^2}\Bigg|_{\substack{P_1=P_1^* \\ P_2=P_2^*}} = \frac{d^2 P_1(x)}{dx^2}\bullet A_{12}\bullet P_2(y)^T \Bigg|_{\substack{x=\frac{1}{2} \\ y=\frac{1}{2}}} = 0,$$

$$B_1 = \frac{\partial^2 u_1(P_1,P_2)}{\partial P_1 \partial P_2}\Bigg|_{\substack{P_1=P_1^* \\ P_2=P_2^*}} = \frac{dP_1(x)}{dx}\bullet A_{12}\bullet \left(\frac{dP_2(y)}{dy}\right)^T \Bigg|_{\substack{x=\frac{1}{2} \\ y=\frac{1}{2}}} = \frac{16}{9}.$$

$D_1 = A_1C_1 - B_1^2 < 0$. Therefore, point (P_1^*, P_2^*) is a saddle point of $u_1(P_1, P_2)$.

Let us examine the solution (P_1^*, P_2^*) for player 2.

$$A_2 = \frac{\partial^2 u_2(P_2,P_1)}{\partial P_2^2}\Bigg|_{\substack{P_1=P_1^* \\ P_2=P_2^*}} = \frac{d^2 P_2(y)}{dy^2}\bullet A_{21}\bullet (P_1(x))^T \Bigg|_{\substack{x=\frac{1}{2} \\ y=\frac{1}{2}}} = 0$$

$$B_2 = \frac{\partial^2 u_2(P_2,P_1)}{\partial P_1 \partial P_2}\Bigg|_{\substack{P_1=P_1^* \\ P_2=P_2^*}} = \frac{dP_2(y)}{dy}\bullet A_{21}\bullet \left(\frac{dP_1(x)}{dx}\right)^T \Bigg|_{\substack{x=\frac{1}{2} \\ y=\frac{1}{2}}} = -\frac{16}{9}.$$

$D_2 = A_2C_2 - B_2^2 < 0$. Thus, point (P_1^*, P_2^*) is a saddle point of $u_2(P_2, P_1)$.

Example 2 - Find mixed NEs in the following 2-player game.

Player II

$$\text{Player I}\quad \begin{array}{c} \\ s_1^1 \\ s_1^2 \\ s_1^3 \end{array}\begin{array}{c} \begin{array}{ccc} s_2^1 & s_2^2 & s_2^3 \end{array} \\ \begin{pmatrix} (1,\ 1) & (3,\ 1) & (2,\ 2) \\ (2,\ 4) & (2,\ 5) & (8,\ 3) \\ (3,\ 3) & (0,\ 4) & (0,\ 1) \end{pmatrix} \end{array}$$

$$A_{12} = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 2 & 8 \\ 3 & 0 & 0 \end{pmatrix},\ A_{21} = \begin{pmatrix} 1 & 4 & 3 \\ 1 & 5 & 4 \\ 2 & 3 & 1 \end{pmatrix}$$

We define two linguistic values $(S, T(S_i), U_i, G_i, M_i)(i \in 2)$, where $U_i = [0,\ 1]$, $T(S_i) = \{s_i^1, s_i^2, s_i^3\}$ (i=1, 2). The semantic rules M_i (i=1, 2) are defined as follows.

$M_1 : T(S_1) \to \{E_1, E_2, E_3\}$,
$M_2 : T(S_2) \to \{H_1, H_2, H_3\}$

where TFNs $E_i, H_i\,(i=1,2,3)$ are defined as follows, $E_1=(0,0,1)$, $E_2=E_3=(0,1,1)$; $H_1=H_2=(0,0,1)$, $H_3=(0,1,1)$. Then, we have $\sum_{i=1}^{3}\mu_i(x)=1+x$ and $\sum_{i=1}^{3}\nu_i(y)=2-y$.

The probability distribution P_i over U_i (i=1, 2) is as follows.

$$P_1(x)=\left(\frac{\mu_1(x)}{\sum_{i=1}^{3}\mu_i(x)},\frac{\mu_2(x)}{\sum_{i=1}^{3}\mu_i(x)},\frac{\mu_3(x)}{\sum_{i=1}^{3}\mu_i(x)}\right)=\left(\frac{1-x}{1+x},\frac{x}{1+x},\frac{x}{1+x}\right),$$

$$P_2(y)=\left(\frac{\nu_1(y)}{\sum_{i=1}^{3}\nu_i(y)},\frac{\nu_2(y)}{\sum_{i=1}^{3}\nu_i(y)},\frac{\nu_3(y)}{\sum_{i=1}^{3}\nu_i(y)}\right)=\left(\frac{1-y}{2-y},\frac{1-y}{2-y},\frac{y}{2-y}\right)$$

Then, we have

$$\frac{dP_1(x)}{dx}=\left(\frac{-2}{(1+x)^2},\frac{1}{(1+x)^2},\frac{1}{(1+x)^2}\right),\quad \frac{dP_2(y)}{dy}=\left(\frac{-1}{(2-y)^2},\frac{-1}{(2-y)^2},\frac{2}{(2-y)^2}\right)$$

One can solve the following system of linear equations.

$$\begin{cases}\dfrac{\partial u_1(P_1,P_{-1})}{\partial P_1}=K_1\bullet(-2,\ 1,\ 1)\bullet\begin{pmatrix}1&3&2\\2&2&8\\3&0&0\end{pmatrix}\bullet\begin{pmatrix}1-y\\1-y\\y\end{pmatrix}=0\\ \dfrac{\partial u_2(P_2,P_{-2})}{\partial P_2}=K_2\bullet(-1,\ -1,\ 2)\bullet\begin{pmatrix}1&4&3\\1&5&4\\2&3&1\end{pmatrix}\bullet\begin{pmatrix}1-x\\x\\x\end{pmatrix}=0\end{cases}$$

where $K_1=\dfrac{1}{(1+x)^2(1+y)}$ and $K_2=\dfrac{1}{(1+y)^2(1+x)}$.

The solution is $x=\frac{1}{5}\in[0,\ 1]$, $y=\frac{1}{5}\in[0,\ 1]$. This 2-player game has a mixed NE (P_1^*, P_2^*) with probability distribution $P_1^*=(\frac{2}{3},\frac{1}{6},\frac{1}{6})$ and $P_2^*=(\frac{4}{9},\frac{4}{9},\frac{1}{9})$.

Let us examine this solution for player 1.

$$A_1=\frac{\partial^2 u_1(P_1,P_2)}{\partial P_1^2}\Bigg|_{\substack{P_1=P_1^*\\P_2=P_2^*}}=\frac{d^2P_1(x)}{dx^2}\bullet A_{12}\bullet P_2(y)^T\Bigg|_{\substack{x=\frac{1}{5}\\y=\frac{1}{5}}}=0$$

$$B_1=\frac{\partial^2 u_1(P_1,P_2)}{\partial P_1\partial P_2}\Bigg|_{\substack{P_1=P_1^*\\P_2=P_2^*}}=\frac{dP_1(x)}{dx}\bullet A_{12}\bullet\left(\frac{dP_2(y)}{dy}\right)^T\Bigg|_{\substack{x=\frac{1}{5}\\y=\frac{1}{5}}}=-\frac{4375}{2916}$$

$D_1=A_1C_1-B_1^2<0$. Therefore, point (P_1^*,P_2^*) is a saddle point of $u_1(P_1,P_2)$.

Let us examine the solution for player 2.

$$A_2=\frac{\partial^2 u_2(P_2,P_1)}{\partial P_2^2}\Bigg|_{\substack{P_1=P_1^*\\P_2=P_2^*}}=\frac{d^2P_2(x)}{dy^2}\bullet A_{21}\bullet P_1(y)^T\Bigg|_{\substack{x=\frac{1}{5}\\y=\frac{1}{5}}}=0$$

$$B_2 = \frac{\partial^2 u_2(P_2,P_1)}{\partial P_1 \partial P_2}\Bigg|_{\substack{P_1=P_1^* \\ P_2=P_2^*}} = \frac{dP_2(y)}{dy} \bullet A_{21} \bullet \left(\frac{dP_1(x)}{dx}\right)^T \Bigg|_{\substack{x=\frac{1}{5} \\ y=\frac{1}{5}}} = -\frac{1250}{486}$$

$D_2 = A_2C_2 - B_2^2 < 0$. Thus, point (P_1^*, P_2^*) is a saddle point of $u_2(P_2, P_1)$.

Example 3 - Find mixed NEs for the following bi-matrix game.

Player II

Player I

$$\begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \end{array}\begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \begin{pmatrix} (0,0) & (1,\text{-}1) & (1,1) & (\text{-}1,0) \\ (\text{-}1,1) & (0,1) & (1,0) & (0,0) \\ (1,0) & (\text{-}1,\text{-}1) & (0,1) & (\text{-}1,1) \\ (1,\text{-}1) & (\text{-}1,0) & (1,\text{-}1) & (0,0) \\ (1,\text{-}1) & (0,0) & (\text{-}1,\text{-}1) & (0,0) \end{pmatrix} \end{array}$$

$$A_{12} = \begin{pmatrix} 0 & 1 & 1 & -1 \\ -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & -1 \\ 1 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix}, \quad A_{21} = \begin{pmatrix} 0 & 1 & 0 & -1 & -1 \\ -1 & 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

We build two linguistic values $(S_i, T_i(S_i), U_i, G_i, M_i)(i \in 2)$. $T_1(S_1) = (s_1, s_2, s_3, s_4, s_5)$ on $U_1 = [0,\ 1]$. $T_2(S_2) = (a_1, a_2, a_3, a_4)$ on $U_2 = [0,\ 1]$, and the semantic rule M_i is defined as follows.
$M_1 : T_1(S_1) = (s_1, s_2, s_3, s_4, s_5) \rightarrow (E_1, E_2, E_3, E_4, E_5)$
$M_2 : T_2(S_2) = (a_1, a_2, a_3, a_4) \rightarrow (H_1, H_2, H_3, H_4)$
where $E_i\,(i = 1,2,3,4,5)$ and $H_i\,(i = 1,2,3,4)$ are TFNs defined in U_1 and U_2 as follows.

$E_1 = E_2 = (0,0,1)$, $E_3 = E_4 = E_5 = (0,1,1)$ and $H_1 = (00,1)$, $H_2 = H_3 = H_4 = (0,1,1)$. Then, we have $\sum_{i=1}^{5} \mu_i(x) = 2 + x$ and $\sum_{i=1}^{4} \nu_i(y) = 1 + 2y$. The probability distribution P_i over U_i (i=1, 2) is as follows.

$$P_1(x) = \left(\frac{\mu_1(x)}{\sum_{i=1}^{5}\mu_i(x)}, \frac{\mu_2(x)}{\sum_{i=1}^{5}\mu_i(x)}, \frac{\mu_3(x)}{\sum_{i=1}^{5}\mu_i(x)}, \frac{\mu_4(x)}{\sum_{i=1}^{5}\mu_i(x)}, \frac{\mu_5(x)}{\sum_{i=1}^{5}\mu_i(x)}\right) = \left(\frac{1-x}{2+x}, \frac{1-x}{2+x}, \frac{x}{2+x}, \frac{x}{2+x}, \frac{x}{2+x}\right),$$

$$P_2(y) = \left(\frac{\nu_1(y)}{\sum_{i=1}^{4}\nu_i(y)}, \frac{\nu_2(y)}{\sum_{i=1}^{4}\nu_i(y)}, \frac{\nu_3(y)}{\sum_{i=1}^{4}\nu_i(y)}, \frac{\nu_4(y)}{\sum_{i=1}^{4}\nu_i(y)}\right) = \left(\frac{1-y}{1+2y}, \frac{y}{1+2y}, \frac{y}{1+2y}, \frac{y}{1+2y}\right)$$

Then, we have

$$\frac{dP_1(x)}{dx} = \left(\frac{-3}{(2+x)^2}, \frac{-3}{(2+x)^2}, \frac{2}{(2+x)^2}, \frac{2}{(2+x)^2}, \frac{2}{(2+x)^2}\right),$$
$$\frac{dP_2(y)}{dy} = \left(\frac{-3}{(1+2y)^2}, \frac{1}{(1+2y)^2}, \frac{1}{(1+2y)^2}, \frac{1}{(1+2y)^2}\right).$$

One can solve the following system of linear equations.

$$\begin{cases} \dfrac{\partial u_1(P_1,P_2)}{\partial P_1} = K_1(-3, \ -3, \ 2, \ 2, \ 2) \bullet \begin{pmatrix} 0 & 1 & 1 & -1 \\ -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & -1 \\ 1 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix} \bullet \begin{pmatrix} 1-y \\ y \\ y \\ y \end{pmatrix} = 0 \\ \dfrac{\partial u_2(P_2,P_{-2})}{\partial P_2} = K_2(-3, \ 1, \ 1 \ 1) \bullet \begin{pmatrix} 0 & 1 & 0 & -1 & -1 \\ -1 & 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \bullet \begin{pmatrix} 1-x \\ 1-x \\ x \\ x \\ x \end{pmatrix} = 0 \end{cases}$$

where $K_1 = \dfrac{1}{(2+x)^2(1+2y)}$ and $K_2 = \dfrac{1}{(1+2y)^2(2+x)}$.

The solution is $x = \dfrac{2}{7} \in [0, \ 1]$, $y = \dfrac{3}{7} \in [0, \ 1]$. This 2-player game has a mixed NE (P_1^*, P_2^*) with probability distribution $P_1^* = (\dfrac{5}{16}, \dfrac{5}{16}, \dfrac{2}{16}, \dfrac{2}{16}, \dfrac{2}{16})$ and $P_2^* = (\dfrac{4}{13}, \dfrac{3}{13}, \dfrac{3}{13}, \dfrac{3}{13})$.

Let us examine this solution (P_1^*, P_2^*) for player 1.

$$A_1 = \frac{\partial^2 u_1(P_1,P_2)}{\partial P_1^2} \Bigg|_{\substack{P_1=P_1^* \\ P_2=P_2^*}} = \frac{d^2 P_1(x)}{dx^2} A_{12} P_2(y)^T \Bigg|_{\substack{x=\frac{2}{3} \\ y=\frac{3}{7}}} = 0$$

$$B_1 = \frac{\partial^2 u_1(P_1,P_2)}{\partial P_1 \partial P_2} \Bigg|_{\substack{P_1=P_1^* \\ P_2=P_2^*}} = \frac{dP_1(x)}{dx} A_{12} \left(\frac{dP_2(y)}{dy}\right)^T \Bigg|_{\substack{x=\frac{2}{3} \\ y=\frac{3}{7}}} = -\frac{93639}{43264}.$$

$D_1 = A_1 C_1 - B_1^2 < 0$. Thus, point (P_1^*, P_2^*) is a saddle point of $u_1(P_1, P_2)$.

Let us examine this solution (P_1^*, P_2^*) for player 2.

$$A_2 = \frac{\partial^2 u_2(P_2,P_1)}{\partial P_2^2} \Bigg|_{\substack{P_1=P_1^* \\ P_2=P_2^*}} = \frac{d^2 P_2(y)}{dy^2} \bullet A_{21} \bullet P_1(x)^T \Bigg|_{\substack{x=\frac{2}{3} \\ y=\frac{3}{7}}} = 0$$

$$B_2 = \frac{\partial^2 u_2(P_2,P_1)}{\partial P_1 \partial P_2} \Bigg|_{\substack{P_1=P_1^* \\ P_2=P_2^*}} = \frac{dP_2(y)}{dy} \bullet A_{21} \bullet \left(\frac{dP_1(x)}{dx}\right)^T \Bigg|_{\substack{x=\frac{2}{3} \\ y=\frac{3}{7}}} = \frac{2401}{2704}$$

$D_2 = A_2 C_2 - B_2^2 < 0$. Therefore, point (P_1^*, P_2^*) is a saddle point of $u_2(P_2, P_1)$.

Conclusion

This article describes an algorithm for calculating mixed NEs in 2-player games. It is proved that the proposed algorithm can calculate mixed NEs in 2-player games within polynomial time. We show that computing mixed NE in any types of bi-matrix games is equivalent to solving a system of 2 linear equations. We claim that the problem of finding mixed NEs in 2-player games is in P-complete class. This algorithm also provides a method to examine the found mixed NE is either a saddle point or a local maximum point of expected payoff function of 2-player games.

Future study will conduct to apply the algorithm to dynamic game theory.

REFERENCES

1. T. Basar and G. J. Olsder, "Dynamic Noncooperative Game Theory", 2nd edition, SIAM, 1995.
2. Bellman R. E. and L. A. Zadeh, Decision making in a fuzzy environment, *Management science*, 17(1970), pp.141-164.

3. V. D. Blondel and J. N. Tsitsiklis, "A Survey of Computational Complexity Results", *Automatica.,* 36 (2000), pp. 1249-1274, Elsevier
4. F. Brandt, F. Fischer and M. Holzer, "Symmetries and Complexity of Pure Nash Equilibrium", *Journal of Computer and System Sciences*, 2009 Elsevier.
5. D. Butnariu, "Fuzzy Games: A Description of Time Concept", *Fuzzy Sets and Systems*, Vol. 1, pp. 181–192, 1978.
6. A. Chakeri and F. Sheikholeslam, "Fuzzy Nash Equilibriums in Crisp and Fuzzy Games", IEEE Transactions on Fuzzy Systems, 2012 IEEE
7. X. Chen and X. Deng, “3-Nash is PPAD-complete”, ECCC, TR05-134, 2005
8. X. Chen, X. Deng and S. Teng, “Settling the Complexity of Computing two-player Nash Equilibria”, *Journal of the ACM*, Vol. 56, Issue 3, 2009
9. X. Chen, D. Durfee, and A. Orfanou, "On the Complexity of Nash Equilibria in Anonymous Games", at arXiv:1412.5681vl, Dec. 2014
10. X. Chen, D. Paparas, and M. Yannakakis, "The complexity of non-monotone markets", In Proceeding of the 45th Annual ACM Symposium on Theory of Computing, pp.181-190, 2013
11. S. Cheng, D. Reeves, Y. Vorobeychik and M. Wellman, "Notes on Equilibria in Symmetric Games", In the Proceeding of the 6th International workshop on Game-Theoretic and Decision-Theoretic Agents, 2004.
12. L. Cigler and B. Faltings, "Symmetric Subgame-Perfect Equilibria in Resource Allocation", *Journal of Artificial Intelligence Research*, 49, pp.323-361, 2014
13. C. Daskalakis, "An efficient PTAS for two-strategy anonymous games", In Proceedings of the 4th International Workshop on Internet and Network economics, pp.186-197, 2008
14. C. Daskalakis, P. W. Goldberg, C. H. Papadimitriou, “The Complexity of Computing a Nash Equilibrium”, ECCC, TR05-115, 2005.
15. C. Daskalakis, and C. H. Papadimitriou, “Three-Player Games are Hard”, ECCC, TR05-139, 2005
16. C. Daskalakis and C. H. Papadimitriou, "Computing equilibria in anonymous games", In Proceedings of the 48th Annual IEEE Symposium on Foundations of Computer Science, pp.83-93, 2007
17. C. Daskalakis and C. H. Papadimitriou, "Discretized multinomial distributions and Nash equilibira in anonymous games" In Proceedings of the 49th Annual IEEE Symposium on Foundations of Computer Science, pp.25-34, 2008
18. C. Daskalakis and C. H. Papadimitriou, "On oblivious PTAS's for Nash equilibrium", In Proceedings of the 4th Annual ACM Symposium on Theory of Computing, pp.75-84, 2009
19. L. Gao, “Finding Nash Equilibria for Polymatrix Games”, International Journal of Computer Science and Artificial Intelligence”, accepted by the editors
20. L. Gao, “A Direct Algorithm for Computing Nash Equilibriums”, International Journal of Computer Science and Artificial Intelligence, Vol. 3, Iss. 2, pp. 80-86, 2013
21. L. Gao, “The Discussion of Applications of the Fuzzy Average to Matrix Game Theory”, The Proceeding of CCECE2012, May 2012.
22. L. Gao, “The Fuzzy Arithmetic Mean”, Fuzzy Sets and Systems, 107, No. 3 pp. 335-348, 1999
23. D. Garagic and J. B. Cruz, Jr, "An Approach to Fuzzy Noncooperative Nash Games", *Journal of Optimization Theory and Applications*, Vol. 118, No.3, pp. 475-491, 2003
24. P. W. Goldberg and A. Turchetta, "Query Complexity of Approximate Equilibria in Anonymous Games", arXiv:1412.6455v2, Apr. 2015
25. A. M. Hefti, "Uniqueness and Stability in Symmetric Games: Theory and Applications", ECON working paper, 2013
26. J. Horner, N. Klein, and S. Rady, "Strongly Symmetric Equilibria in Brandit Games", 2013 https://www.economicdynamics.org/meetpapers/2013/paper_1107.pdf
27. A. Kaufmann, and M. M. Gupta, *Fuzzy Mathematical Models in Engineering and Management Science*, Elsevier, Amsterdam, 1998.
28. R. Mehta, V. V. Vazirani and S. Yazdanbod, “On Settling Some Open Problems on 2-Player Symmetric Nash Equilibria”, The Proceeding of 8th International Symposium on Algorithmic Game Theory, SAGT2015
29. J. Nash, "Non-Cooperative Games", The Annals of Mathematics, Vol. 54, Iss. 2 pp286-295, 1951.
30. Von Neumann, Morgenstern, *Theory of Games and Economic Behavior*, Princeton University Press, 1944
31. R. Porter, E. Nudelman and Y. Shohem, “Simple Search Methods for Finding a Nash Equilibrium”, *Games and Economic Behavior*, 63 (2008), pp642-662 Elsevier
32. C. T. Ryan, A. Jiang and K. L. Brown, "Computing Pure Strategy Nash Equilibria in Compact Symmetric Games", The Proceeding of 11th ACM Conference on EC, June 7-11,2010.
33. W. H. Sandholm, “Lecture Notes on Game Theory”, 2014, at www.ssc.wisc.edu/~whs/teaching/711/lngt.pdf
34. W. H. Sandholm, "Evolutionary Game Theory", at www.ssc.wisc.edu/~whs/research/edt.pdf
35. S. H. Wu, and V. W. Soo, "A Fuzzy Theoretic Approach to Multi-Agent Coordination, Multiagent Platforms, Springer Verlag, New York, NY, pp. 76–87, 1998.
36. L. A. Zadeh, “The Concept of Linguistic Variable and its Application to Approximate Reasoning-1”, Information Sciences, 8, pp.199-249, 1975
37. H. J. Zimmermann, *Fuzzy Set Theory and Its Application*, 2nd edition, Kluwer Academic Publishers, Boston, 1991.

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