ISSN: 0975-833X

Available online at http://www.journalcra.com

International Journal of Current Research
Vol. 5, Issue, 03, pp.424-430, March, 2013

INTERNATIONAL JOURNAL OF CURRENT RESEARCH

RESEARCH ARTICLE

THE BONDAGE NUMBERS EXTENDED TO DIRECTED CIRCULAR-ARC GRAPHS

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ARTICLE INFO

Article History:
Received 28th December, 2012
Received in revised form
11th January, 2013
Accepted 23rd February, 2013
Published online 19th March, 2013

Key words:
Circular-arc family, Circular-arc graph, Dominating set, Domination number, Bondage number, Directed graph, in-degree, out-degree, In-neighbor and out-neighbor.

ABSTRACT

Circular-arc graphs are rich in combinatorial structures and have found applications in several disciplines such as Biology, Ecology, Genetics, Computer Science and particularly useful in cyclic scheduling. Dominating sets play predominant role in the theory of graphs. In this paper we consider the bondage number $b(G)$ for a Circular-arc family A and G is a Circular-arc graph corresponding to arcs A which is defined as the minimum number of edges whose removal results in a new graph with larger domination number. Among the various applications of the theory of domination the most often discussed is a communication network. This network consists of communication links between a fixed set of sites. By constructing a family of minimum dominating sets, we compute the bondage number $b(G) \leq d(v) + d^{-}(u) - \left|N^{-}(u) \cap N^{-}(v)\right|$. Suppose, communication network fails due to link failure. Then the problem is to find a fewest number of communication links such that the communication with all sites in possible. This leads to the introducing of the concept of bondage number of graph.

INTRODUCTION

Let $A = \{A_1, A_2, ..., A_n\}$ be a Circular-arc family on a Circle. Where each A_i is an arc. Without loss of generality assume that the end points of all arcs are distinct and no arc covers the entire Circle. Denote an arc i that begins at p and ends at point q in the clockwise direction by (p, q). Define p to be the head and q to be the tail of the arc i and now i is denoted by $i = (p, q)$. Two arcs j and i are said to intersect each other if they have non-empty intersection.

Let $G(V, E)$ be a graph. Let $A = \{A_1, A_2, ..., A_n\}$ be a family of arcs on a Circle. Then G is called a Circular-arc graph, if there is a one-to-one correspondence between V and A such that two vertices in V are adjacent if and only if their corresponding arcs in A intersect [9,10]. It is well known that the topological structure of an interconnection network can be modeled by a connected graph whose vertices represent sites of the network and whose edges represent physical communication links. A subset D of V is said to be a dominating set of G if every vertex in $V \setminus D$ is adjacent to a vertex in D. The bondage number $b(G)$ of a non – empty graph G is the minimum cardinality among all sets of edges E_1 for which $\gamma(G - E_1) > \gamma(G)$ [2]. Here $\gamma(G)$ indicates the domination number of G. Thus, the bondage number of G is the smallest number of edges whose removal makes every minimum dominating set in G a non – dominating set in the resultant spanning sub graph [5]. Since the domination number of every spanning sub graph of a non – empty graph G is at least as great as $\gamma(G)$, the bondage number of a non – empty graph is well defined[3,4,6,7]. This concept was introduced by Fink et.al [8] and they have studied this parameter for some standard graphs, trees and general bounds are obtained. A directed graph or digraph is a graph each of whose edges has a direction [1]. For $v \in V$ and $(u, v), (v, w) \in E$, u and w are called an *in-neighbor* and an *out-neighbor* of v, respectively. The *in-degree* and the *out-degree* of v are the number of its in-neighbors and out-neighbors, denoted by $d^{-}(v)$ and $d^{+}(v)$, respectively. The *degree* of v is $d(v) = d^{+}(v) + d^{-}(v)$. In this connection we should not consider reverse direction of the directions of the circular-arc graphs.

MAIN THEOREMS

Theorem 1

Let $A = \{1, 2, ..., n\}$ be a circular arc family and G be a circular arc graph corresponding to the circular arc family A. Let $i, j \in A$ and suppose j is contained in i and there is no other arc that intersects j other than i, then $b(G) = 1$, it gives,

$$b(G) \leq d(v) + d^-(u) - \left|N^-(u) \cap N^-(v)\right|$$

Proof

Let $A = \{1, 2, ..., n\}$ be the given circular arc family. Let G be the circular arc graph corresponding to the given arc family A. Let i, j be any two arcs in A which satisfy the hypothesis of the theorem. Then clearly $i \in D$ where D is a minimum dominating set of G, because there is no other arc in A other than i, that dominates j. We consider the edge $e = (i, j)$ in G. If we remove this edge from G, then j becomes an isolated vertex in $G - e$, as there is no other vertex in G, other than i that is adjacent with j. Now the dominating set $D_1 = D \cup \{j\}$ becomes a dominating set of $G - e$. Since, D is a minimum dominating set of G as well as D_1 is also a minimum dominating set of $G - e$. Therefore $\gamma(G-e) = |D_1| = |D| + 1 > |D| = \gamma(G)$. Thus $b(G) = 1$ Similarly, we will prove that as said above $b(G) \leq d(v) + d^-(u) - \left|N^-(u) \cap N^-(v)\right|$ First we will discuss the directed graph corresponding to an interval graph. A digraph with a Vertex-Set V and an Edge-Set E.

For a Subset $S \subseteq V$, let

$$\left.\begin{array}{l} E^+(S) = \{(u,v) \in E(G) : u \in S, v \notin S\} \\ E^-(S) = \{(u,v) \in E(G) : u \notin S, v \in S\} \\ N^+(S) = \{v \in V : u \in S, (u,v) \in E^+(S)\} \\ N^-(S) = \{u \in V : v \in S, (u,v) \in E^-(S)\} \end{array}\right\} ------(I)$$

Now, We will prove the bondage number $b(G)$. Consider the following Circular- arc family

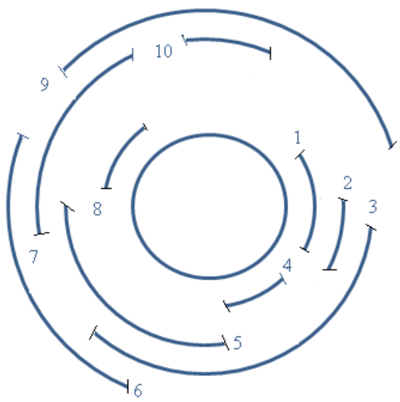

Figure 1Cicular-arc family A

From A, we have nbd [1] = {1,2,3,9}, nbd [2] = {1,2,3}, nbd [3] = {1,2,3,4,5,6,}, nbd [4] = {3,4}, nbd [5] = {3,5,6,7}, nbd [6] = {3,5,6,7,8}, nbd [7] = {5,6,7,8,9}, nbd [8] = {6,7,8,9}, nbd [9] = {1,7,8,9,10}, nbd [10] = {9,10}

Dominating set $D = \{3, 9\}$ and $\gamma(G) = 2$, Remove the edge $e = (3, 4)$ from G, Dominating set of $G - e = D_1 = \{3, 4, 9\}$ and $\gamma(G - e) = 3$. Therefore $\gamma(G - e) > \gamma(G)$ and hence $b(G) = 1$. Now, we will prove the following inequality, $b(G) \leq d(v) + d^-(u) - \left|N^-(u) \cap N^-(v)\right|$ Let us consider the vertices, $u = 3, v = 4$ Such that $(u, v) = (3, 4) \in E(G)$

Here in this interval family clearly $(3, 4) \in E(G)$ Now $d(v) = d(4)$ $d(v) =$ out degree of v + in degree of v

$$\therefore d(v) = d^+(v) + d^-(v)$$
$$= d^+(4) + d^-(4)$$
$$= 0 + 1$$
$$= 1$$
$$\therefore d(v) = 1 --------> (1)$$
$$\Rightarrow d^-(u) = d^-(1) = 1$$
$$\therefore d^-(u) = 1 --------> (2)$$

Now to find $N^{-}(u)$ and $N^{-}(v)$, we need to find $E^{-}(S)$, Where $S \subseteq V$, the vertex set of G

Where, $E^{-}(S) = \{(u,v) \in E(G) : u \notin S, v \in S\}$

Now let us take,

$$S = \{3,4\} \Rightarrow E^{-}(3) = \{(1,3),(2,3)\} \Rightarrow E^{-}(4) = \{\phi\}$$

Now,

$$N^{+}(S) = \{v \in V : u \in S, (u,v) \in E^{+}(S)\}$$

$$N^{-}(S) = \{u \in V : v \in S, (u,v) \in E^{-}(S)\} ---------- > (3)$$

From equation (3)

$$\Rightarrow N^{-}(u) = N^{-}(3) = \{1,2\} \Rightarrow N^{-}(u) = \{1,2\}$$

$$\Rightarrow N^{-}(v) = N^{-}(4) = \{\phi\} \Rightarrow N^{-}(v) = \{\phi\}$$

$$\Rightarrow N^{-}(u) \cap N^{-}(v) = \{\phi\}$$

$$\Rightarrow \left|N^{-}(u) \cap N^{-}(v)\right| = 0 -------------- > (4)$$

Hence finally, from $(1),(2)$ and (4)

$$d(v) + d^{-}(u) - \left|N^{-}(u) \cap N^{-}(v)\right| = 1 + 1 - 0 = 2, b(G) = 1$$

$\therefore b(G) \le d(v) + d^{-}(u) - \left|N^{-}(u) \cap N^{-}(v)\right|$ is proved.

Theorem 2

Let $A = \{1,2,.....,n\}$ be a circular arc family corresponding to a Circular arc graph G. Let the dominating set D of G consists of two vertices only, say x and y. Suppose x dominates the vertex set $S_1 = \{1,2,.....,i\}$ and y dominates the vertex set $S_2 = \{i+1,.....,n\}$. If there is no vertex in S_1 other than x that dominates S_1 and no vertex in S_2 other than y that dominates S_2. Then the bondage number $b(G) = 1$, as well as it gives

$$b(G) \le d(v) + d^{-}(u) - \left|N^{-}(u) \cap N^{-}(v)\right|$$

Proof

Let the dominating set $D = \{x, y\}$. Suppose, x and y satisfy the hypothesis of the theorem, since x alone dominates S_1, there is no vertex in $S_3 = \{1,2,.....,i\}/\{x\}$ that can dominates S_2. Let j be any vertex in S_3 and $e = (x, j)$. We consider the graph $G-e$. In this graph, x dominates every vertex in S_1 except j. Now consider a vertex in S_1 which is adjacent with j ,say k, then clearly the set $\{x,k\}$ dominates the set S_1 in $G-e$. If there is no vertex in S_1 that is adjacent with j, then clearly the graph G becomes disconnected. So there is at least one vertex in S_1 that is adjacent with j. Let us assume that there is a single vertex say z, $z \ne x$ such that z dominates the set S_1 in $G-e$. This implies that z also dominates the set S_1 in G, a contradiction, because by hypothesis x is the only vertex that dominates the set S_1 in G. Hence a single vertex cannot dominate S_1 in $G-e$. Thus $D_1 = D \cup \{k\}$ becomes a dominating set of $G-e$. Where D is minimum in G, D_1 is also minimum in $G-e$. So that $\gamma(G-e) > \gamma(G)$. Hence $b(G) = 1$. A similar argument with vertex y also gives $b(G) = 1$, our aim is to prove that $b(G) \le d(v) + d^{-}(u) - \left|N^{-}(u) \cap N^{-}(v)\right|$

Let us consider the Circular-arc family $A = \{1,2,.....,8\}$ and choose $S_1 = \{1,2,3,4\}, S_2 = \{5,6,7,8\}$

Since nbd [1] = {1,2,7}, nbd [2] = {1,2,3,4}, nbd [3] = {2,3,4}, nbd [4] = {2,3,4,5}, nbd [5] = {4,5,6}, nbd [6] = {5,6,7,8}, nbd [7] = {1,6,7,8}, nbd [8] = {6,7,8}

From (I) We will prove, the bondage number $b(G)$.

Dominating set of $G = D = \{2,6\}$ and $\gamma(G) = 2$

Remove the edge $e = (2,4)$ from G

Dominating set of $G - e = D_1 = \{2,3,6\}$ & $\gamma(G-e) = 3$

Therefore $\gamma(G-e) > \gamma(G)$ and thus $b(G) = 1$.

Now, we will prove the following inequality,

$$b(G) \le d(v) + d^-(u) - |N^-(u) \cap N^-(v)|$$

Let us consider the vertices, $u = 2, v = 3$ Such that $(u,v) = (2,3) \in E(G)$

Here in this interval family clearly $(u,v) = (2,3) \in E(G)$

$d(v) = d(2)$

$\Rightarrow d(v) = d^+(v) + d^-(v)$

$\Rightarrow d(v) = d^+(2) + d^-(2)$

$\Rightarrow d(v) = 2+1$

$\Rightarrow d(v) = 3$

$\therefore d(v) = 3 --------> (1)$

$\Rightarrow d^-(u) = d^-(1) = 1$

$\therefore d^-(u) = 1 ------> (2)$

Now, to find $N^-(u)$ and $N^-(v)$ we need to find $E^-(S),$ Where $S \subseteq V$, the vertex set of G

Where $E^-(S) = \{(u,v) \in E(G) : u \notin S, v \in S\}$

Now let us take, $S = \{2,3\}$

$\Rightarrow E^-(2) = \{(1,2)\}$

$\Rightarrow E^-(3) = \{\phi\}$

Now,

$N^+(S) = \{v \in V : u \in S, (u,v) \in E^+(S)\}$

$N^-(S) = \{u \in V : v \in S, (u,v) \in E^-(S)\} --------> (3)$

From Equation (3)

$\Rightarrow N^-(u) = N^-(2) = \{1\} \Rightarrow N^-(u) = \{1\}$

$\Rightarrow N^-(v) = N^-(3) = \{\phi\} \Rightarrow N^-(v) = \{\phi\}$

$\Rightarrow N^-(u) \cap N^-(v) = \{\phi\}$

$\Rightarrow |N^-(u) \cap N^-(v)| = 0 -----------> (4)$

Hence finally, from $(1), (2)$ and (4)

$d(v) + d^-(u) - |N^-(u) \cap N^-(v)| = 3+1-0 = 4$ and $b(G) = 1$ $\therefore$

$b(G) \le d(v) + d^-(u) - |N^-(u) \cap N^-(v)|$

Theorem 3

Let $A = \{1, 2, ..., n\}$ be a circular arc family corresponding to a Circular arc graph G. Let the dominating set D of G consists of two vertices only say x and y. Suppose x dominates the vertex set $S_1 = \{1, 2, ..., i\}$ and y dominates the vertex set $S_2 = \{i+1, ..., n\}$. Suppose there is one more vertex $m \in S_1$ or S_2 respectively, which dominates S_1 or S_2 then the bondage number $b(G) = 1$, provided there is no backword arc in the dominating set D, it leads to

$$b(G) \le d(v) + d^-(u) - |N^-(u) \cap N^-(v)|$$

Proof

Let $A = \{1, 2, ..., n\}$ be a circular arc family, and G be a circular arc graph corresponding to A. Let the dominating set $D = \{x, y\}$ and x dominates S_1 and y dominates S_2. Let $m \in S_1$ such that m also dominates S_1. Let the edge $e = (x, m)$. We consider the graph $G - e$. In this graph the vertices x and m are not adjacent. Hence x alone cannot dominate the set S_1.

We require at least two vertices in S_1, which dominate S_1 in $G - e$.Therefore the dominating set of $G - e$ contains more than two vertices. Thus $\gamma(G - e) > \gamma(G)$. Hence the bondage number $b(G) = 1$, similar is the case if $m \in S_2$.

The above as follows, the procedure of our aim bondage number

$$b(G) \le d(v) + d^-(u) - \left|N^-(u) \cap N^-(v)\right|$$

Let us consider $A = \{1, 2, \ldots\ldots, 10\}, S_1 = \{1, 2, 3, 4, 5\}$ and $S_2 = \{6, 7, 8, 9, 10\}$

nbd [1] = {1,2,3,10}, nbd [2] = {1,2,3,4,5}, nbd [3] = {1,2,3,4,5}, nbd [4] = {2,3,4,5}, nbd [5] = {2,3,4,5,6}, nbd [6] = {5,6,7,8,9}, nbd [7] = {6,7,8,9}, nbd [8] = {6,7,8,9,10}, nbd [9] = {6,7,8,9,10}, nbd [10] = {1,8,9,10}

From (I) We will prove, the bondage number $b(G)$.
Dominating set of $G = D = \{2,8\}$ and $\gamma(G) = 2$
Remove the edge $e = (2,3)$ from G Dominating set of $G - e = D_1 = \{2,3,8\}$ and $\gamma(G-e) = 3$ Therefore $\gamma(G-e) > \gamma(G)$ and thus $b(G) = 1$. Now, we will prove the following inequality,

$$b(G) \le d(v) + d^-(u) - \left|N^-(u) \cap N^-(v)\right|$$

Let us consider the vertices, $u = 2, v = 3$ Such that $(u, v) = (2,3) \in E(G)$ Here in this interval family clearly $(u, v) = (2,3) \in E(G)$

$$d(v) = d(3)$$
$$\Rightarrow d(v) = d^+(v) + d^-(v)$$
$$\Rightarrow d(v) = d^+(3) + d^-(3)$$
$$\Rightarrow d(v) = 2 + 2$$
$$\Rightarrow d(v) = 4$$
$$\therefore d(v) = 4 - - - - - - - - - > (1)$$
$$\Rightarrow d^-(u) = d^-(2) = 1$$
$$\therefore d^-(u) = 1 - - - - - - - > (2)$$

Now, to find $N^-(u)$ and $N^-(v)$ we need to find $E^-(S)$, Where $S \subseteq V$, the vertex set of G
Where $E^-(S) = \{(u,v) \in E(G) : u \notin S, v \in S\}$
Now let us take, $S = \{2,3\}$

$$\Rightarrow E^-(2) = \{(1,2)\}$$
$$\Rightarrow E^-(3) = \{(1,3)\}$$

Now,

$$N^+(S) = \{v \in V : u \in S, (u,v) \in E^+(S)\}$$
$$N^-(S) = \{u \in V : v \in S, (u,v) \in E^-(S)\} - - - - - - - - - > (3)$$

From Equation (3)

$$\Rightarrow N^-(u) = N^-(2) = \{1\} \Rightarrow N^-(u) = \{1\}$$
$$\Rightarrow N^-(v) = N^-(3) = \{1\} \Rightarrow N^-(v) = \{1\}$$
$$\Rightarrow N^-(u) \cap N^-(v) = \{1\}$$
$$\Rightarrow \left|N^-(u) \cap N^-(v)\right| = 1 - - - - - - - - - - - - > (4)$$

Hence finally, from $(1), (2)$ and (4)

$$d(v) + d^-(u) - \left|N^-(u) \cap N^-(v)\right| = 4 + 1 - 1 = 4,\ b(G) = 1$$

$$\therefore\ b(G) \le d(v) + d^-(u) - \left|N^-(u) \cap N^-(v)\right|$$

Theorem4

Let the Circular arc family $A = \{1, 2, \ldots\ldots, n\}$ and G be a Circular arc graph corresponding to A. Let $D = \{x, y\}$. Suppose x dominates $S_1 = \{1, 2, \ldots\ldots, i\}$ and y dominates $S_2 = \{i+1, \ldots\ldots, n\}$. Suppose there are two vertices say $m_1, m_2 \in S_1$ or S_2 such that $m_1, m_2 \in S_1$ also dominates S_1 or S_2 respectively, then the bondage number $b(G) = 3$ as well as it gives

$$b(G) \le d(v) + d^-(u) - \left|N^-(u) \cap N^-(v)\right|$$

Proof

Let $D=\{x,y\}$ and x,y satisfy the hypothesis of the theorem. Suppose $m_1,m_2\in S_1$ and m_1,m_2 also dominate S_1. Let l be an arbitrary vertex in S_1, $l\neq i,x,m_1,m_2$. Now delete the edges xl,m_1l,m_2l that are incident with l from G. If $d(l)=3$, then l becomes an isolated vertex in $G_1=G-\{xl,m_1l,m_2l\}$. Thus $D_1=D\cup\{l\}$ becomes a dominating set of G_1 and since D is minimum it follows that D_1 is also minimum in G_1. Therefore $\gamma(G_1)>\gamma(G)$ and hence $b(G)=3$ .Suppose $d(l)>3$. Then there is atleast one vertex, say j in S_1 such that j is adjacent to l and $j\neq x,m_1,m_2$. Let $G_1=G-\{xl,m_1l,m_2l\}$ .In G_1, l is not dominated by x,m_1,m_2, but is dominated by j. Further every vertex in S_1 otherthan l is dominated by x or m_1 or m_2 in G_1. Therefore every vertex in S_1 is dominated by $\{x,j\}$ or $\{m_1,j\}$ or $\{m_2,j\}$ in G_1. Thus $D_1=D\cup\{j\}$ becomes a dominating set of G_1 and since D is minimum in G, it follows that D_1 is also minimum in G_1 Hence $\gamma(G_1)>\gamma(G)$ so that $b(G)=3$. Similar is the case if $m_1,m_2\in S_2$.

The above as follows, the procedure of our aim bondage number

$$b(G)\leq d(v)+d^-(u)-\left|N^-(u)\cap N^-(v)\right|$$

Let us consider $A=\{1,2,.....,11\}$, $S_1=\{1,2,3,4,5,6\}$ and $S_2=\{7,8,9,10,11\}$

nbd [1] = {1,2,3,4,11}, nbd [2] = {1,2,3,4,5,6}, nbd [3] = {1,2,3,4,5,6}, nbd [4] = {1,2,3,4,5,6}, nbd [5] = {2,3,4,5,6}, nbd [6] = {2,3,4,5,6,7}, nbd [7] = {6,7,8,9,10}, nbd [8] = {7,8,9,10}, nbd [9] = {7,8,9,10,11}, nbd [10] = {7,8,9,10,11}, nbd [11] = {1,9,10,11}

From (I) We will prove, the bondage number $b(G)$.

Dominating set of $G=D=\{3,9\}$ and $\gamma(G)=2$ Remove the edges $(3,5),(4,5),(2,5)$ from G Dominating set of $G-e=D_1=\{3,5,9\}\ \&\ \gamma(G_1)=3$ Therefore $\gamma(G_1)>\gamma(G)$ and thus $b(G)=3$ Now, we will prove the following inequality,

$$b(G)\leq d(v)+d^-(u)-\left|N^-(u)\cap N^-(v)\right|$$

Let us consider the vertices, $u=3, v=5$ Such that $(u,v)=(3,5)\in E(G)$

Here in this interval family clearly $(u,v)=(3,5)\in E(G)$

$$d(v)=d(5)$$
$$\Rightarrow d(v)=d^+(v)+d^-(v)$$
$$\Rightarrow d(v)=d^+(5)+d^-(5)$$
$$\Rightarrow d(v)=1+3$$
$$\Rightarrow d(v)=4$$
$$\therefore d(v)=4--------->(1)$$
$$\Rightarrow d^-(u)=d^-(3)=2$$
$$\therefore d^-(u)=2------->(2)$$

Now, to find $N^-(u)$ and $N^-(v)$ we need to find $E^-(S)$, Where $S\subseteq V$, the vertex set of G

Where $E^-(S)=\{(u,v)\in E(G): u\notin S, v\in S\}$

Now let us take, $S=\{3,5\}$

$$\Rightarrow E^-(3)=\{(1,3),(2,3\}$$
$$\Rightarrow E^-(3)=\{(2,5),(4,5\}$$

Now,

$$N^+(S)=\{v\in V: u\in S,(u,v)\in E^+(S)\}$$
$$N^-(S)=\{u\in V: v\in S,(u,v)\in E^-(S)\}--------->(3)$$

From Equation (3)

$\Rightarrow N^{-}(u) = N^{-}(3) = \{1,2\} \Rightarrow N^{-}(u) = \{1,2\}$

$\Rightarrow N^{-}(v) = N^{-}(5) = \{2,4\} \Rightarrow N^{-}(v) = \{2,4\}$

$\Rightarrow N^{-}(u) \cap N^{-}(v) = \{2\}$

$\Rightarrow \left|N^{-}(u) \cap N^{-}(v)\right| = 1 ----------- > (4)$

Hence finally, from $(1),(2)$ and (4)

$d(v) + d^{-}(u) - \left|N^{-}(u) \cap N^{-}(v)\right| = 4 + 2 - 1 = 5$

And $b(G) = 3$

$\therefore b(G) \le d(v) + d^{-}(u) - \left|N^{-}(u) \cap N^{-}(v)\right|$

Theorem5:

Let $A = \{1,2,.....,n\}$ be a circular-arc family and $D = \{x, y, z\}$. Suppose x dominates $S_1 = \{1,2,.....,i\}$, y dominates $S_2 = \{i+1,....., j\}$ and z dominates $S_3 = \{j+1,.....,n\}$. If there are two other vertices in S_1 or S_2 or S_3 that dominates the sets respectively, then the bondage number $b(G) = 1$ as

$b(G) \le d(v) + d^{-}(u) - \left|N^{-}(u) \cap N^{-}(v)\right|$

Proof:

The proof is similar to that of Theorem 2.

Theorem6:

Let $D = \{x, y, z\}$. Suppose x dominates $S_1 = \{1,2,.....,i\}$, y dominates $S_2 = \{i+1,....., j\}$ and z dominates $S_3 = \{j+1,.....,n\}$. If there is one more vertex $m \in S_1$ or S_2 or S_3, that dominates S_1 or S_2 or S_3 respectively. Then the bondage number $b(G) = 1$, occurs

$b(G) \le d(v) + d^{-}(u) - \left|N^{-}(u) \cap N^{-}(v)\right|$

Proof

The proof is similar to that of Theorem 3.

Theorem7

Let $D = \{x, y, z\}$ .Suppose x dominates $S_1 = \{1,2,.....,i\}$, y dominates $S_2 = \{i+1,....., j\}$ and z dominates $S_3 = \{j+1,.....,n\}$ .Suppose there are two vertices say $m_1, m_2 \in S_1$ or S_2 or S_3, such that m_1, m_2 also dominate S_1 or S_2 or S_3 respectively. Then the bondage number $b(G) = 3$ as well as, it gives

$b(G) \le d(v) + d^{-}(u) - \left|N^{-}(u) \cap N^{-}(v)\right|$

Proof

The proof is similar to that of Theorem 4.

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