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International Journal of Current Research
Vol. 3, Issue, 11, pp.371-374, October, 2011

INTERNATIONAL JOURNAL OF CURRENT RESEARCH

ISSN: 0975-833X

RESEARCH ARTICLE

CERTAIN RESULTS IN A SINGLE SERVER QUEUEING SYSTEM

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ARTICLE INFO

Article History:
Received 08th July, 2011
Received in revised form
19th August, 2011
Accepted 27th September, 2011
Published online 30th October, 2011

Key words:
Birth and Death Process,
M/M/1 queue,
Departure Process.

ABSTRACT

This paper presents certain results pertaining to a single server queue with Poisson arrivals and exponential service times. A closed form solution is obtained for the probability that, exactly i arrivals and j services occur over a time interval of length t in a queuing system that is idle at the beginning of the interval. Section 1.1 describes the model. Solution of the model is obtained in Section 1.2. By using $P_n(t)$, performance measures are obtained in Section 1.3.

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INTRODUCTION

The queues that are emptied and restarted periodically and thus not subjected to analysis using the celebrated equilibrium results, there are several potential applications for results obtained. The classical queuing results can be obtained as a special case of the general solution. The classical analysis of the M / M / 1 queue is based on an analysis where the state of the system is defined as the number of units in the system, including the one being serviced. The system is treated as a birth-death process in which arrivals cause the state to increase by one and departures cause the state to decrease by one. The differential-difference equations for the time dependent probability functions P_n (t), the probability that there are exactly n customers in the system at epoch *t*, may be written directly as

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu P_1(t) \qquad for\, n=0$$

$$\frac{dP_n(t)}{dt} = \lambda P_{n-1}(t) - (\lambda+\mu)\; P_n(t) + \mu\, P_{n+1}(t) \qquad for\; n > 0$$

For a complete derivation of these equations one may refer to Gross *et al.* (2008). Unfortunately, the solution to these equations is quite complex, involving modified Bessel functions and infinite series of such functions. It may be observed that these Bessel functions can be approximated and the expression for the solution simplified to a considerable extent. That is to say that approximate solutions for the equations can be numerically obtained by limiting to steady-state solution. That is

$$P_n = (\lambda/\mu)^n \; (1 - (\lambda/\mu))$$

derived by solving the equations obtained by setting $dP_n(t)$ / dt = 0 for all n. The P_n is then used to derive the expected queue length, average time spent in the system and utilization of the server for steady – state operation. In several potential applications of queuing theory, the system never reaches equilibrium (See Borovkov (1999)). The system begins operation empty and then stops or is stopped at some point of time t. Businesses or service operations such as barber shops or physicians' offices, which open and close never, operate under steady state conditions. The classical transient results for the M / M /1 queue provide an insight into the behavior of a queuing system through a fixed operation time t. The functions $P_n(t)$ gives the distribution for the number in the system at time *t*, but provide virtually no information on how the system has operated up until time t? One may like to know what will happen up to time t? Typical questions include. (i) How many customers will be processed by time t? and (ii) What fraction of time will the server be busy during the first t time units of operation? (See Meyn and Tweedie(1993) Kannan and Lakshmikantham (2001), Ross (2006) and Gross *et al.* (2008). Moreover, if the system begins operation empty, the fraction of time the server is busy, the expected queue length, and the initial rate of output from the system will be below the steady-state values so that the use of steady-state results to obtain these measures is not appropriate. Section 1.1 describes the model. The solution of the model is obtained in Section 1.2. Using $P_n(t)$ various performance measures have been expressed in Section 1.3.

DESCRIPTION OF THE MODEL

In contrast to the classical development, we base our analysis of the M / M / 1 queues on a model in which the state of the system is given by (i, j), where i is the number of arrivals and j is the number of departures until time t. We denote the state probabilities for the model as P_{ij} (t). P_{ij} (t) is the probability that exactly i arrivals and j services have occurred by time t (See Ross(1980), Bhat (2008) and Grimval *et al.* (2010)). The solution for P_{ij}(t) provides considerable information concerning the transient behavior of the queue. Some performance measures that can be expressed in terms of P_{ij} (t) are discussed later in this paper. Now we proceed to obtain P_{ij} (t). It can be verified that P_{ij} (t) satisfies the differential-difference equations given below.

$$\frac{dP_{\infty}(t)}{dt} = -\lambda P_{\infty}(t) \qquad (1.1.1)$$

$$\frac{dP_{\infty}(t)}{dt} = \lambda P_{i-1,0}(t) - (\lambda+\mu)\ P_{i0}(t) \qquad for \quad i \ge 1, \qquad (1.1.2)$$

$$\frac{dP_{\infty}(t)}{dt} = \mu P_{i,i-1}(t) - \lambda P_{ii}(t) \qquad for \quad i \ge 1, \qquad (1.1.3)$$

$$\frac{dP_{\infty}(t)}{dt} = \mu P_{i,j-1}(t) + \lambda P_{i-1,j}(t) - (\lambda+\mu)P_{ij}(t) \qquad for \quad i \ge 2 \text{ and } 1 \le j \le i \qquad (1.1.4)$$

SOLUTION OF THE MODEL

From (1.1.1) to (1.1.4) the general form for the Laplace transform can be written as

$$P_{ij}(s) = \int_0^{\infty} e^{-st}\ P_{ij}(t)\ dt$$

Then we have,

$$p_{ij}(s) = \left(\frac{\lambda}{\lambda+\mu+s}\right)^i \left(\frac{\mu}{\lambda+s}\right)^j \sum_{k=0}^{j} \frac{(i-k)(i+k-1)!}{k!i!} \frac{(\lambda+s)^{k-1}}{(\lambda+\mu+s)^k}$$

for $i \ge 1$ *and* $0 \le j \le i$ (1.2.1)

Further,

$$P_{\infty}(s) = (\lambda+s)^{-1} \qquad (1.2.2)$$

This result can be obtained by induction and then show that, P_{ij} (t) satisfies the system of different of differential – difference equations for the model. We shall verify this by induction. The general form for p_{ij} (s) given as (1.2.1) can be verified by induction. To accomplish the induction it is sufficient to show the following results:

R-1: Equation (1.2.1) is true for p_{10} (s), p_{20} (s) and p_{11} (s)

R-2: If (1.2.1) is assumed to be true for p_{10} (s), then it is true for $p_{i+1.0}$ (s)

R-3: If (1.2.1) is true for $p_{i-1,j}$ (s), and $p_{i,j-1}$ (s) then it is true for p_{ij} (s) where i > j.

R-4: If (1.2.1) is true for $p_{i.,i-1}$ (s), then it is true for p_{ii} (s),

R-1: is accomplished by observing that, according to (1.2.1),

$$P_{10}(s) = \lambda(\lambda+\mu+s)^{-1}(\lambda+s)^{-1}$$

$$P_{20}(s) = \lambda^2(\lambda+\mu+s)^{-2}(\lambda+s)^{-1}$$

$$P_{11}(s) = \lambda(\lambda+\mu+s)^{-1}\mu(\lambda+s)^{-2}$$

and that these results agree with those obtained directly by taking transforms of (1.1.2) and (1.1.3). R-2: can be demonstrated by a simple induction on R-1. We know from R-1: that (1.2.1) is true for p_{10} (s). From (1.1.2), we can see that $p_{i+1,0}$ (s) satisfies the following relation:

$s\, p_{i+1,0}(s) = \lambda\, p_{i,0}(s) - (\lambda+\mu)\, p_{i+1,0}(s)$ or

$$p_{i+1,0}(s) = \lambda(\lambda+\mu+s)^{-1}\ p_{i,0}(s)$$

Using (1.2.1) we obtain

$$p_{i+1,0}(s) = \lambda\ (\lambda+\mu+s)^{-1} \left(\frac{\lambda}{\lambda+\mu+s}\right)^i (\lambda+s)^{-1} = \left(\frac{\lambda}{\lambda+\mu+s}\right)^{i+1} (\lambda+s)^{-1}$$

which agrees with $p_{i+1,0}$ (s) as given by (1.2.1)

For R-3, we note from (1.1.4) that, for i > j p_{ij} (s) satisfies the following relation

$s\, p_{ij}(s) = \mu\ p_{i,j-1}(s) + \lambda\, p_{i-1,j}(s) - (\lambda+\mu)\, p_{ij}(s)$ or

$$p_{ij}(s) = \frac{\mu}{\lambda+\mu+s}\ p_{i,j-1}(s) + \frac{\lambda}{\lambda+\mu+s}\ p_{i-1j}(s)$$

Using (1.2.1) on the right hand side of this equation, we obtain

$$p_{ij}(s) = \frac{\mu}{\lambda+\mu+s}\left(\frac{\lambda}{\lambda+\mu+s}\right)^i \left(\frac{\mu}{\lambda+s}\right)^{j-1} \sum_{k=0}^{j-1} \frac{(i-k)(i+k-1)!}{k!1!} \frac{(\lambda+s)^{k-1}}{(\lambda+\mu+s)^k} + \left(\frac{\lambda}{\lambda+\mu+s}\right)\left(\frac{\lambda}{\lambda+\mu+s}\right)^{i-1} \left(\frac{\mu}{\lambda+s}\right)^j M(j)$$

where $$M(j) = \sum_{k=0}^{j} \frac{(i-k-1)(i+k-2)!}{k!(i-1)!} \frac{(\lambda+s)^{k-1}}{(\lambda+\mu+s)^k}$$

$$= \left(\frac{\lambda}{\lambda+\mu+s}\right)^i \left(\frac{\mu}{\mu+s}\right)^j \left\{\sum_{k=0}^{j-1} \frac{(i-k)(i+k-1)!}{k!i!} \frac{(\lambda+s)^k}{(\lambda+\mu+s)^{k+1}} + M(j)\right\}$$

$$= \left(\frac{\lambda}{\lambda+\mu+s}\right)^i \left(\frac{\mu}{\mu+s}\right)^j \left\{\sum_{k=0}^{j-1} \frac{(i-k)(i+k-1)!}{k!i!} \frac{(\lambda+s)^k}{(\lambda+\mu+s)^{k+1}} + M(j)\right\}$$

$$= \left(\frac{\lambda}{\lambda+\mu+s}\right)^i \left(\frac{\mu}{\mu+s}\right)^j \left\{\sum_{k=1}^{j} \frac{(i-k-1)(i+k-2)!i}{i!k!} \frac{(\lambda+s)^{k-1}}{(\lambda+\mu+s)^k} + (i+k)\right\} + \frac{1}{\lambda+s}$$

$$= \left(\frac{\lambda}{\lambda+\mu+s}\right)^i \left(\frac{\mu}{\mu+s}\right)^j \left\{\frac{1}{\lambda+s} \sum_{k=1}^{j} \frac{(i-k)\ (i+k-1)\ (i+k-2)!}{i!\ k!} \frac{(\lambda+s)^{k-1}}{(\lambda+\mu+s)^k}\right\}$$

$$= \left(\frac{\lambda}{\lambda+\mu+s}\right)^i \left(\frac{\mu}{\lambda+s}\right)^j \left\{\frac{1}{\lambda+s} + \sum_{k=1}^{j} \frac{(i-k)(i+k-1)(i+k-2)!}{i!k!} \frac{(\lambda+s)^{k-1}}{(\lambda+\mu+s)^k}\right\}$$

$$= \left(\frac{\lambda}{\lambda+\mu+s}\right)^i \left(\frac{\mu}{\lambda+s}\right)^j \sum_{k=0}^{j} \frac{(i-k)\ (i+k-1)!}{i!k!} \frac{(\lambda+s)^{k-1}}{(\lambda+\mu+s)^k}$$

which is the expression for p_{ij} (s) according to (1.2.1).

The demonstration of R-4: is accomplished by observing that (1.1.3) requires that

$$p_{u}(s) = \frac{\mu}{\lambda+s}\, p_{i,i-1}(s)$$

Substitution for $p_{i,i-1}(s)$ according to (1.2.1) yields

$$p_{ii}(s) = \frac{\mu}{\lambda+s}\left(\frac{\lambda}{\lambda+\mu+s}\right)^{i}\left(\frac{\mu}{\lambda+s}\right)^{i-1}\sum_{k=0}^{i-1}\frac{(i-k)(i+k-1)}{k!\,i!}\frac{(\lambda+s)^{k-1}}{(\lambda+\mu+s)^{k}}$$

$$=\left(\frac{\lambda}{\lambda+\mu+s}\right)^{i}\left(\frac{\mu}{\lambda+s}\right)^{i}\sum_{k=0}^{i}\frac{(i-k)\,(i+k-1)}{k!\,i!}\frac{(\lambda+s)^{k-1}}{(\lambda+\mu+s)^{k}}$$

since the term in the sum is zero when k = i. This result for $p_{ii}(s)$ agrees with $p_{ij}(s)$ according to (1.2.1). The proof that (1.2.1) represents the general form for $p_{ij}(s)$ is thus complete. We shall now move to determine $p_{ij}(t)$ by determining the inverse Laplace transform of $p_{ij}(s)$. Rearranging the expression for $p_{ij}(s)$ we obtain

$$p_{ij}(s) = \frac{\lambda^{i}\mu^{j}}{i!}\sum_{k=0}^{j}\frac{(i-k)(i+k-1)!}{k!}\frac{1}{(\lambda+s)^{j-k+1}(\lambda+\mu+s)^{i+k}} \qquad (1.2.3)$$

The inverse transform of $f(s) = 1/(s+a)^{n}$ is $F(t) = t^{n-1}e^{-at}/(n-1)!$ Using this result and observing that the inverse of $f_1(s)*f_2(s)$ is the convolution of $F_1(t)$ and $F_2(t)$, we obtain

$$p_{ij}(t) = \frac{\lambda^{i}\mu^{j}}{i!}\sum_{k=0}^{j}\frac{(i-k)\,(i+k-1)!}{k!}\,[(j-k)!\,(i+k-1)!]^{-1}\int_{0}^{t}(t-\tau)^{j-k}e^{-\lambda(t-\tau)}\tau^{i+k-1}e^{-(\lambda+\mu)\tau}\,d\tau \qquad (1.2.4)$$

Substituting w = τ / t yields

$$P_{ij}(t) = \lambda^{i}\mu^{j}e^{-\lambda t}t^{(i+j)}\sum_{k=0}^{j}\frac{(i-k)}{i!\,k!\,(j-k)!\,i!}\int_{0}^{t}(1-w)^{j-k}w^{i+k-1}e^{-\mu w}\,dw \qquad (1.2.5)$$

Replacing $(1-w)^{j-k}$ by its binomial expansion we obtain

$$P_{ij}(t) = \frac{\lambda^{i}\mu^{j}e^{-\lambda t}t^{(i+j)}}{i!}\sum_{k=0}^{j}\frac{(i-k)}{k!\,(j-k)!}\sum_{m=0}^{j-k}\frac{(-1)^{m}(j-k)!}{(j-k-m)!\,m!}\int_{0}^{t}e^{-\mu w}w^{m+i+k-1}\,dw \qquad (1.2.6)$$

where the integral is of the form

$$\int_{0}^{1}e^{-ax}x^{n}\,dx = \frac{n!}{a^{n+1}}\left[1-e^{-a}\sum_{r=0}^{n}\frac{a^{r}}{r!}\right]$$

Substituting for the integral and re-arranging we get

$$P_{ij}(t) = \left(\frac{\lambda}{\mu}\right)^{i}\frac{(\mu t)^{j}e^{-\lambda t}}{i!}\sum_{k=0}^{j}\frac{(i-k)}{k!}$$

for i ≥ 1 and 0 ≤ j ≤ i (1.2.7)

which is the required result. The special case for i = 0 and j = 0 is obtained by directly solving (1.1.1).

$P_{\infty}(t) = e^{-\lambda t}$ which provides the solution for $P_{ij}(t)$

For various values of λ, μ, t and n numerically one can verify that (1.2.6) is valid as (1.2.8) is satisfied. Thus we see that

$$P\,(n\ arrivals\ in\,(0,t)) = \frac{e^{-\lambda}\,(\lambda t)^{n}}{n!} = \sum_{j=0}^{\infty}P_{nj}(t) \qquad (1.2.8)$$

CERTAIN MEASURES OF PERFORMANCE

The departure process from the M / M /1 queue has the distribution function $P_{.j}(t)$ the probability that exactly j customers have been served by time t. In terms of $P_{ij}(t)$ we have

$$P_{.j}(t) = \sum_{i=j}^{\infty}P_{ij}(t)$$

The density function of T, the time for the j th departure, can be obtained as

$$f_{T}(t) = \sum_{k=j}^{\infty}\frac{dP_{.k}(t)}{dt} = -\sum_{k=0}^{j-1}\frac{dP_{.k}(t)}{dt}$$

since

$$P(T \le t) = P(\text{at least } j \text{ departures in } (0, t)) = \sum_{k=j}^{\infty}P_{.k}(t)$$

The probability of exactly n customers in the system at time t, denoted by $P_{n}(t)$, can be expressed in terms $P_{ij}(t)$ as

$$P_{n}(t) = \sum_{j=0}^{\infty}P_{j+n,j}(t)$$

Although results for $P_{n}(t)$ are available from the classical one dimension state model, the expression above is computationally simpler since the infinite sum does not involve Bessel functions explicitly. The waiting time distribution for a customer can be derived as P (W > τ | t), the probability that a customer waits more than τ time units in the system, given that the customer arrives at time t.

$$P(W>\tau \mid t) = \sum_{n=0}^{\infty}P(W>\tau \mid n \text{ customers in system at time t})\,P_{n}(t)$$

$$=\sum_{n=0}^{\infty}P(\text{number of services by time } \tau < n+1)\,P_{n}(t)$$

$$=e^{-\mu\tau}\sum_{n=0}^{\infty}\sum_{s=0}^{n}\frac{(\mu\tau)^{s}}{s!}P_{n}(t)$$

Therefore, the cumulative distribution for the sojourn time in the system is

$$P\ (W \leq \tau \mid t) = 1 - e^{-\mu\tau} \sum_{n=0}^{\infty} \sum_{s=0}^{n} \frac{(\mu\tau)^s}{s!} P_n\ (t)$$

The density function of sojourn time for a customer arriving at time t is obtained by differentiating this expression with respect to τ. The system utilization, i.e., that fraction of time

P_{ij} (t). The probability that the system is empty at some time τ is $\sum_{j=0}^{\infty} P_{jj}(\tau)$. Thus the fraction of the time that the system is empty and consequently the server is idle [I(t)]. Thus,

$$I\ (t) = \frac{1}{t} \int_0^t \sum_{j=0}^{\infty} P_{jj}\ (\tau)\ d\tau$$

and the fraction of time that the system is non-empty and, hence, the server utilized

[$U_{(t)}$] is

$$U_{(t)} = 1 - \frac{1}{t} \int_0^t \sum_{j=0}^{\infty} P_{jj}(\tau)\ d\tau$$

Other measures such as the auto-correlation of the departure process and the cross-correlation at the arrival and departure process can also be obtained. They will be functions of both time t and the interval τ for which the correlation coefficient can be calculated.

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