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RESEARCH ARTICLE

NUMERICAL INVESTIGATION OF THE HEAT FLOW PROBLEM USING RAYLEIGH RITZ, STWS AND RUNGE-KUTTA METHODS BASED ON VARIOUS MEANS

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ABSTRACT

In this article numerical investigation of an interesting heat flow problem is discussed using Rayleigh Ritz, single-term Walsh series (STWS) method and Runge-Kutta (RK) method based on various means. The results (approximate solutions) obtained very accurate using the above said methods are compared with the exact solution of that problem. It is found that the solution obtained using RKCeM (Runge-Kutta Centroidal Mean) is closer to the exact solution of the heat flow problem. The high accuracy and the wide applicability of RKCeM approach will be demonstrated with numerical example. Solution graphs for discrete exact solutions are presented in a graphical form to show the efficiency of the RKCeM. The results obtained show that RKCeM is more useful for solving the heat flow problem and the solution can be obtained for any length of time.

INTRODUCTION

A mathematical model is adescription of a system using mathematical concepts and language. The process of developing a mathematical model is termed mathematical modelling. Mathematical models are usednotonlyinthe natural sciences (suchas physics, biology, earth science, meteorology) and engineering disciplines (e.g. computer science, artificial intelligence), but also in the social sciences (such as economics, psychology, sociology and political science), physicists, engineers, statisticians, operations research analysts and economists use mathematical models most extensively. Mathematical models can take many forms, including but not limited to dynamical systems, statistical models, differential equations, or game theoretic models. These and other types of models can overlap, with a given model involving a variety of abstract structures. In general, mathematical models may include logical models, as far as logic is taken as a part of mathematics. In many cases, the quality of a scientific field depends on how well the mathematical models developed on the theoretical side agree with results of repeatable experiments. Lack of agreement between theoretical mathematical models and experimental measurements often leads to important advances as better theories are developed. The STWS method and extended fourth order RK methods found wide applications in the field of optimum control of linear systems with quadratic index, signal processing, electronic circuits and singular non-linear systems [1- 7]. In this article, new methods are introduced to solve the unsteady one-dimensional heat-flow problem via ODE, which involve two phases. In phase-I, the spatial dependency of the heat flow equation is eliminated by applying the Rayleigh-Ritz method and to determine the suitable initial conditions, the Galerkin Technique is utilized. In phase II, the resulting system of equations is being solved by applying the methods STWS, RKCeM and RKHaM (Runge-Kutta Harmonic Mean) to determine the discrete solutions of the unsteady heat- flow problem. Further, to analyze the efficiency of the above-mentioned methods, the discrete solutions obtained are compared with the exact solutions and with the obtained discrete solutions by the methods of Laplace Transform and RKAM (Runge-Kutta Arithmetic Mean).

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PROBLEM AND SOLUTION

Let us consider an unsteady one-dimensional heat flow problem (it may be referred as a flow of electricity in cables – the telegraph problem). The governing equation of the flow is given by

$$\frac{\partial T}{\partial t} = \alpha^2 \frac{\partial^2 T}{\partial x^2}, \quad 0 < x < 1 \tag{1}$$

where T denotes the temperature, t denotes the time, α^2 denotes the thermal diffusivity and x denotes the space coordinate.

The initial and boundary conditions are

$$T(x,0) = 1.0 \tag{2}$$

and

$$T(0,t) = \frac{\partial T}{\partial x}(1,t) = 0 \tag{3}$$

PHASE I

RAYLEIGH - RITZ METHOD

This method is used for the elimination of spatial dependency in eq. (1). Assuming that T^* is the weighting function of T, which satisfies the initial and boundary conditions given by eqs. (2) and (3), the following weighted residual equation can be obtained as (Schechter [8])

$$\int_0^1 T^* \left[\frac{\partial T}{\partial t} - \alpha^2 \frac{\partial^2 T}{\partial x^2} \right] dx = 0 \tag{4}$$

After integrating and introducing the boundary conditions (3) we obtain

$$\int_0^1 T^* \frac{\partial T}{\partial t} dx + \alpha^2 \int_0^1 \frac{\partial T^*}{\partial x} \frac{\partial T}{\partial x} dx = 0 \tag{5}$$

Assuming the same function has been applied for T and T^*, then we define

$$T = \sum_{j=1}^{2} C_j(t)\phi_j(x) \tag{6}$$

$$T^* = \sum_{k=1}^{2} C_k(t)\phi_k(x) \tag{7}$$

where $\phi_1 = x$ and $\phi_2 = x^2$. Substituting eqs. (6) and (7) into eq. (5) we obtain

$$\int_0^1 \phi_k \left[\sum_{j=1}^{2} \frac{\partial C_j}{\partial t} \phi_j \right] dx + \alpha^2 \int_0^1 \left[\sum_{\substack{k=1 \\ j=1}}^{2} C_j \frac{\partial \phi_k}{\partial x} \frac{\partial \phi_j}{\partial x} \right] dx = 0 \tag{8}$$

Eq. (8) can be expressed as

$$A\,C'(t) + \alpha^2 B\,C(t) = 0 \tag{9}$$

where

$$A = \int_0^1 \phi_k\ \phi_j\ dx, \qquad B = \int_0^1 \frac{\partial \phi_k}{\partial x} \frac{\partial \phi_j}{\partial x} dx,$$

$$C'(t) = \left[C_1'(t) \quad C_2'(t) \right] \text{ and } C(t) = \left[C_1(t) \quad C_2(t) \right]$$

Evaluating the indicated integration, we get

$$\begin{bmatrix} 20 & 15 \\ 15 & 12 \end{bmatrix} \begin{bmatrix} C_1'(t) \\ C_2'(t) \end{bmatrix} + \alpha^2 \begin{bmatrix} 60 & 60 \\ 60 & 80 \end{bmatrix} \begin{bmatrix} C_1(t) \\ C_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{10}$$

GALERKIN METHOD

To solve the system, we need some initial conditions for C_1 and C_2, since in the present approximation, the initial condition T(X, 0) = 1 cannot be satisfied. We then represent the residual of the approximation with the initial condition as (Schechter [8])

$$E_1 = T(x,0) - 1 = xC_1(0) + x^2C_2(0) - 1 \tag{11}$$

Now, employing the Galerkin method, we get

$$\int_0^1 \left[x\,C_1(0) + x^2 C_2(0) - 1 \right] x\,dx = 0 \tag{12}$$

$$\int_0^1 \left[x\,C_1(0) + x^2 C_2(0) - 1 \right] x^2\,dx = 0 \tag{13}$$

Solving eqs. (12) and (13), we obtain

$$C_1(0) = 4, \quad C_2(0) = -10/3 \tag{14}$$

Hence, for the problem (1), the spatial dependency of the heat flow has been eliminated by applying the Rayleigh-Ritz method thereby reducing the problem to a system of linear first order differential equations (10) whose initial conditions are given in (14).

PHASE II - COMPUTATION OF $C_1(t)$ AND $C_2(t)$

Here, numerical methods namely STWS and fourth order RK methods based on AM, CeM and HaM, have been introduced to calculate $C_1(t)$ and $C_2(t)$ for the system (10).

SINGLE TERM WALSH SERIES (STWS) TECHNIQUE

Consider the system of linear differential equations

$$K\,x'(t) = A\,x(t) + B\,u(t) \tag{15}$$

with $x(0) = x_0$.

where K and A are n x n matrices, B is an n x r matrix, x(t) is an n-state vector, and u(t) is an r-input vector. In this technique, the given function is expanded as a single - term Walsh series in the normalized interval $\tau \in [0.1)$, which corresponds to $t \in [0.1/m)$ by defining $t = \tau/m$, m being any integer. The following are the recursive relations, in STWS method, to determine the discrete solution for the system (15).

$$R_i = \left[K - \frac{A}{2m} \right]^{-1} S_i$$

$$P_i = \frac{R_i}{2} + x(i-1) \tag{16}$$

$$x(i) = R_i + x(i-1)$$

where $S_i = \frac{A}{m} x(i-1) + \frac{B}{m} u_i$; **i = 1, 2, 3, ….**

Then, x(i) will give the discrete values of the state and P_i gives the Block Pulse Function (BPF) values of the state to any length of time. The main advantage of this method is that if the matrix K in (15) is singular, this difference $\left[K - \frac{A}{2m} \right]$ turns out to be non-singular. Hence, the inverse of the matrix can be computed.

The state – space equation (10) is

$$\begin{bmatrix} 20 & 15 \\ 15 & 12 \end{bmatrix} \begin{bmatrix} C_1'(t) \\ C_2'(t) \end{bmatrix} + \alpha^2 \begin{bmatrix} 60 & 60 \\ 60 & 80 \end{bmatrix} \begin{bmatrix} C_1(t) \\ C_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad 17)$$

with $C(0) = [C_1(0)\ \ C_2(0)]^T = [4\ \ -10/3]^T$.

Table 1. Variation of T(x,t) for $\alpha^2 = 0.5$

Time	Exact	Laplace	RKAM	RKCeM	RKHaM	STWS
Value of x = 0.5						
0.20	0.7354	0.7220	0.7219	0.7219	0.7218	0.7201
0.40	0.5529	0.5555	0.5555	0.5555	0.5554	0.5553
0.60	0.4295	0.4329	0.4329	0.4329	0.4329	0.4328
0.80	0.3353	0.3376	0.3376	0.3376	0.3376	0.3375
1.00	0.2619	0.2633	0.2633	0.2633	0.2633	0.2632
1.20	0.2046	0.2053	0.2054	0.2054	0.2053	0.2053
1.40	0.1598	0.1601	0.1602	0.1602	0.1601	0.1601
Value of x = 1.0						
0.20	0.9488	0.9820	0.9820	0.9821	0.9819	0.9863
0.40	0.7717	0.7843	0.7844	0.7844	0.7843	0.7846
0.60	0.6062	0.6124	0.6125	0.6125	0.6124	0.6123
0.80	0.4739	0.4777	0.4777	0.4777	0.4776	0.4775
1.00	0.3703	0.3725	0.3725	0.3726	0.3725	0.3724
1.20	0.2892	0.2905	0.2905	0.2906	0.2905	0.2904
1.40	0.2260	0.2266	0.2266	0.2266	0.2266	0.2265

Table 2. Variation of T(x,t) for $\alpha^2 = 0.75$

Time	Exact	Laplace	RKAM	RKCeM	RKHaM	STWS
Value of x = 0.5						
0.20	0.6322	0.6306	0.6306	0.6307	0.6304	0.6293
0.40	0.4295	0.4330	0.4329	0.433	0.4328	0.4327
0.60	0.2963	0.2982	0.2982	0.2982	0.2981	0.2979
0.80	0.2046	0.2055	0.2054	0.2054	0.2053	0.2051
1.00	0.1413	0.1414	0.1414	0.1415	0.1414	0.1412
1.20	0.0975	0.0974	0.0974	0.0974	0.0974	0.0973
1.40	0.0673	0.0671	0.0671	0.0671	0.0671	0.0670
Value of x = 1.0						
0.20	0.8637	0.8843	0.8843	0.8844	0.8840	0.8867
0.40	0.6062	0.6124	0.6125	0.6125	0.6122	0.6122
0.60	0.4189	0.4218	0.4219	0.4219	0.4217	0.4215
0.80	0.2892	0.2905	0.2905	0.2906	0.2904	0.2902
1.00	0.1997	0.2001	0.2001	0.2001	0.2000	0.1998
1.20	0.1379	0.1378	0.1378	0.1378	0.1378	0.1376
1.40	0.0952	0.0950	0.0949	0.0949	0.0949	0.0947

Table 3. Variation of T(x,t) for $\alpha^2 = 1.0$

Time	Exact	Laplace	RKAM	RKCeM	RKHaM	STWS
Value of x = 0.5						
0.20	0.5529	0.5555	0.5555	0.5556	0.5551	0.5548
0.40	0.3353	0.3376	0.3376	0.3377	0.3374	0.3372
0.60	0.2046	0.2054	0.2054	0.2054	0.2052	0.2050
0.80	0.1249	0.1249	0.1249	0.1249	0.1248	0.1246
1.00	0.0762	0.0760	0.0760	0.076	0.0759	0.0757
1.20	0.0465	0.0463	0.0462	0.0462	0.0462	0.0460
1.40	0.0284	0.0281	0.0281	0.0281	0.0281	0.0280
Value of x = 1.0						
0.20	0.7717	0.7844	0.7844	0.7846	0.7838	0.7848
0.40	0.4739	0.4777	0.4777	0.4778	0.4774	0.4771
0.60	0.2892	0.2906	0.2905	0.2906	0.2904	0.2900
0.80	0.1765	0.1767	0.1767	0.1768	0.1766	0.1763
1.00	0.1077	0.1075	0.1075	0.1075	0.1074	0.1071
1.20	0.0657	0.0654	0.0654	0.0654	0.0653	0.0651
1.40	0.0401	0.0398	0.0398	0.0398	0.0397	0.0396

Table 4. Variation of T(x,t) for $\alpha^2 = 2.0$

Time	Exact	Laplace	RKAM	RKCeM	RKHaM	STWS
Value of x = 0.5						
0.20	0.3353	0.3376	0.3376	0.3381	0.3363	0.3366
0.40	0.1249	0.1249	0.1249	0.1251	0.1244	0.1236
0.60	0.0465	0.0462	0.0462	0.0463	0.0460	0.0455
0.80	0.0173	0.0171	0.0171	0.0171	0.0170	0.0167
1.00	0.0064	0.0063	0.0063	0.0063	0.0063	0.0062
1.20	0.0024	0.0023	0.0023	0.0023	0.0023	0.0023
1.40	0.0009	0.0009	0.0009	0.0009	0.0009	0.0008
Value of x = 1.0						
0.20	0.4739	0.4777	0.4777	0.4784	0.4758	0.4734
0.40	0.1765	0.1767	0.1767	0.177	0.1760	0.1749
0.60	0.0657	0.0654	0.0654	0.0655	0.0651	0.0644
0.80	0.0245	0.0242	0.0242	0.0242	0.0241	0.0237
1.00	0.0091	0.0090	0.0089	0.009	0.0089	0.0087
1.20	0.0034	0.0033	0.0033	0.0033	0.0033	0.0032
1.40	0.0013	0.0012	0.0012	0.0012	0.0012	0.0012

Table 5 Absolute Error in T(x,t) for $\alpha^2 = 0.5$

	Time	Laplace	RKAM	RKCeM	RKHaM	STWS
When x = 0.5	0.3353	0.0134	0.0134	0.0134	0.0135	0.0135
	0.1249	0.0026	0.0026	0.0026	0.0025	0.0026
	0.0465	0.0034	0.0034	0.0034	0.0033	0.0034
	0.0173	0.0023	0.0023	0.0023	0.0023	0.0023
	0.0064	0.0014	0.0014	0.0014	0.0014	0.0014
	0.0024	0.0007	0.0008	0.0008	0.0007	0.0008
	0.0009	0.0003	0.0003	0.0003	0.0003	0.0003
When x = 1.0	0.3353	0.3376	0.3376	0.3381	0.3363	0.3366
	0.1249	0.1249	0.1249	0.1251	0.1244	0.1236
	0.0465	0.0462	0.0462	0.0463	0.0460	0.0455
	0.0173	0.0171	0.0171	0.0171	0.0170	0.0167
	0.0064	0.0063	0.0063	0.0063	0.0063	0.0062
	0.0024	0.0023	0.0023	0.0023	0.0023	0.0023
	0.0009	0.0009	0.0009	0.0009	0.0009	0.0008

Table 6. Absolute Error in T(x,t) for $\alpha^2 = 0.75$

	Time	Laplace	RKAM	RKCeM	RKHaM	STWS
When x = 0.5	0.20	0.0016	0.0017	0.0016	0.0019	0.0017
	0.40	0.0035	0.0034	0.0034	0.0032	0.0034
	0.60	0.0019	0.0018	0.0019	0.0017	0.0018
	0.80	0.0009	0.0008	0.0008	0.0007	0.0008
	1.00	0.0001	0.0002	0.0002	0.0001	0.0002
	1.20	0.0001	0.0001	0.0001	0.0002	0.0001
	1.40	0.0002	0.0003	0.0002	0.0003	0.0003
When x = 1.0	0.20	0.0206	0.0207	0.0208	0.0204	0.0207
	0.40	0.0062	0.0063	0.0063	0.006	0.0063
	0.60	0.0029	0.003	0.003	0.0028	0.0029
	0.80	0.0013	0.0013	0.0013	0.0012	0.0013
	1.00	0.0004	0.0004	0.0004	0.0003	0.0004
	1.20	0.0001	0.0001	0	0.0001	0.0001
	1.40	0.0002	0.0003	0.0003	0.0003	0.0003

Table 7. Absolute Error in T(x,t) for $\alpha^2 = 1.0$

	Time	Laplace	RKAM	RKCeM	RKHaM	STWS
When x = 0.5	0.20	0.0026	0.0026	0.0208	0.0022	0.0026
	0.40	0.0023	0.0023	0.0063	0.0021	0.0023
	0.60	0.0008	0.0008	0.0030	0.0006	0.0007
	0.80	0.0000	0.0000	0.0013	0.0000	0.0000
	1.00	0.0002	0.0002	0.0004	0.0003	0.0002
	1.20	0.0002	0.0003	0.0000	0.0003	0.0003
	1.40	0.0003	0.0003	0.0003	0.0003	0.0003
When x = 1.0	0.20	0.0127	0.0127	0.0128	0.0121	0.0127
	0.40	0.0038	0.0038	0.0039	0.0034	0.0038
	0.60	0.0014	0.0013	0.0014	0.0011	0.0013
	0.80	0.0002	0.0002	0.0002	0.0001	0.0002
	1.00	0.0002	0.0002	0.0002	0.0003	0.0002
	1.20	0.0003	0.0004	0.0003	0.0004	0.0004
	1.40	0.0003	0.0004	0.0003	0.0004	0.0004

Table 8. Absolute Error in T(x,t) for $\alpha^2 = 2.0$

	Time	Laplace	RKAM	RKCeM	RKHaM	STWS
When x = 0.5	0.20	0.0023	0.0023	0.0028	0.0010	0.0023
	0.40	0.0000	0.0000	0.0002	0.0004	0.0000
	0.60	0.0003	0.0003	0.0002	0.0005	0.0003
	0.80	0.0002	0.0002	0.0002	0.0003	0.0002
	1.00	0.0001	0.0001	0.0001	0.0001	0.0001
	1.20	0.0001	0.0001	0.0001	0.0001	0.0001
	1.40	0.0000	0.0000	0.0000	0.0000	0.0000
When x = 1.0	0.20	0.0038	0.0038	0.0045	0.0019	0.0037
	0.40	0.0002	0.0002	0.0005	0.0005	0.0002
	0.60	0.0003	0.0004	0.0003	0.0006	0.0004
	0.80	0.0003	0.0003	0.0003	0.0004	0.0003
	1.00	0.0001	0.0002	0.0002	0.0002	0.0002
	1.20	0.0001	0.0001	0.0001	0.0001	0.0001
	1.40	0.0000	0.0000	0.0000	0.0000	0.0000

If $A = \begin{bmatrix} 20 & 15 \\ 15 & 12 \end{bmatrix}$ and $B = \begin{bmatrix} 60 & 60 \\ 60 & 80 \end{bmatrix}$

then the eq. (17) becomes

$$A\,C'(t) = -\alpha^2 B\,C(t) \qquad (18)$$

with $C(0) = [\ 4\ \ -10/3]^T$. Applying the STWS approach, the following recursive relationship is obtained.

$$R_i = \left[A + \frac{\alpha^2 B}{2m}\right]^{-1} S_i$$

$$P_i = \frac{R_i}{2} + C(i-1) \qquad (19)$$

$$C(i) = R_i + C(i-1)$$

where $S_i = \frac{\alpha^2}{m}.BC(i-1)$

and i = 1, 2, 3, … the interval number. The discrete and Block Pulse Function (BPF) values of C(t) are obtained from C(i) and P_i, to any length of time. To obtain the discrete solutions, via extended RK methods, we write the system of eqs. (10) explicitly as :

$$C_1' = \alpha^2 [12 C_1 + 32 C_2]$$

$$C_2' = -\alpha^2 \left[20 C_1 + \frac{700}{15} C_2\right] \qquad (20)$$

EXTENDED RUNGE - KUTTA METHOD BASED ON AM

The general p-stage RK method for solving $\dot{x} = f(t, x)$ is defined by

$$x_{n+1} = x_n + h \sum_{i=1}^{p} b_i k_i \text{ where}$$

$$k_i = f\left(t_n + c_i h,\ x_n + h \sum_{i=1}^{p} a_{ij} k_i\right)$$

$$c_i = \sum_{i=1}^{p} a_{ij},\ i = 1, 2, ..., p,$$

where b and c are p-dimensional vectors and the matrix A = (a_{ij}) is of order (p x p). Hence the fourth order RK method for solving an IVP of the form

$$\dot{x} = f(t, x) \text{ with } x(0) = x_0$$

can be formulated as

$$x_{n+1} = x_n + \frac{h}{3} \sum_{i=1}^{3} \left[\frac{K_i + K_{i+1}}{2}\right]$$

i.e.,

$$x_{n+1} = x_n + \frac{h}{3}\left[\frac{k_1 + k_2}{2} + \frac{k_2 + k_3}{2} + \frac{k_3 + k_4}{2}\right]$$

$$x_{n+1} = x_n + \frac{h}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

In the initial iteration, we get

x (1) = x (0) + Δx

where $\Delta x = \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$k_1 = f(t(0), x(0))$$

$$k_2 = f\left(t(0) + \frac{h}{2},\ x(0) + \frac{h}{2} k_1\right)$$

$$k_3 = f\left(t(0) + \frac{h}{2},\ x(0) + \frac{h}{2} k_2\right)$$

$$k_4 = f(t(0) + h,\ x(0) + h k_3)$$

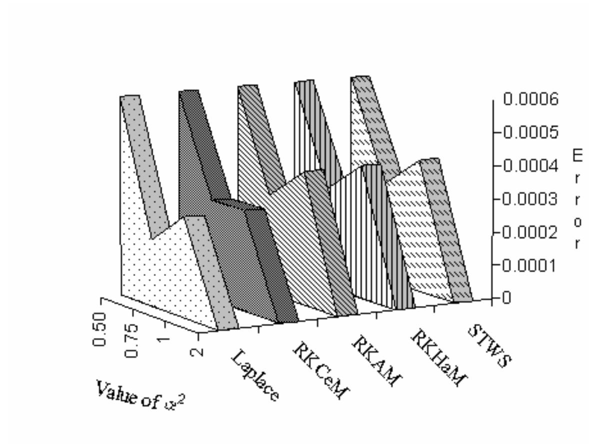

Fig.1. Error graph for x = 1.0 at t = 1.4

EXTENDED RUNGE - KUTTA METHOD BASED ON CEM

In [1 – 3, 10-11], Evans and Yaakub have developed a new RK method of order 4 based on Centroidal mean to solve first order equation and it is to be noted that the Centroidal Mean of two points x_1 and x_2 is defined as

$$\frac{2}{3}\left(\frac{x_1^2 + x_1 x_2 + x_2^2}{x_1 + x_2}\right)$$

Consider the first order equation (2.1) of the form

$$y' = f(x, y)$$

with $y(x_0) = y_0$.

Let h denote the interval between equidistant values of x. The fourth order RKAM formula (2.21) can be written as

$$y_{n+1} = y_n + \frac{h}{3}\left(\frac{k_1 + k_2}{2} + \frac{k_2 + k_3}{2} + \frac{k_3 + k_4}{2}\right)$$

$$y_{n+1} = y_n + \frac{h}{3}\left(\sum_{i=1}^{3} \frac{k_i + k_{i+1}}{2}\right)$$

and substituting the arithmetic mean (AM) of k_i, $1 \le i \le 6$ with their Centroidal Means we obtain a new formula, similar to the above equation, as

$$y_{n+1} = y_n + \frac{h}{3}\left[\sum_{i=1}^{3} \frac{2(k_i^2 + k_i k_{i+1} + k_{i+1}^2)}{3(k_i + k_{i+1})}\right]$$

to obtain the fourth order formula in the form,

$$k_1 = f(x_n, y_n)$$
$$k_2 = f(x_n + a_1 h, y_n + h a_1 k_1)$$
$$k_3 = f(x_n + (a_2 + a_3)h, \; y_n + h a_2 k_1 + h a_3 k_2)$$
$$k_4 = f(x_n + (a_4 + a_5 + a_6)h, \; y_n + h a_4 k_1 + h a_5 k_2 + h a_6 k_3)$$
$$y_{n+1} = y_n + \frac{h}{3}\left[\frac{2(k_1^2 + k_1 k_2 + k_2^2)}{3(k_1 + k_2)} + \frac{2(k_2^2 + k_2 k_3 + k_3^2)}{3(k_2 + k_3)} + \frac{2(k_3^2 + k_3 k_4 + k_4^2)}{3(k_3 + k_4)}\right]$$

that is $y_{n+1} = y_n + \frac{UPPER}{LOWER}$

where, UPPER =

$$\frac{2h}{9}\left[(k_1^2 + k_1 k_2 + k_2^2)(k_2 + k_3)(k_3 + k_4) + (k_2^2 + k_2 k_3 + k_3^2)(k_1 + k_2)(k_3 + k_4)\right.$$
$$\left. + (k_3^2 + k_3 k_4 + k_4^2)(k_1 + k_2)(k_2 + k_3)\right]$$

and,

LOWER = $(k_1 + k_2)(k_2 + k_3)(k_3 + k_4)$,

while the Taylor series expansion of $y(x_n+1)$ may be given as,

TAYLOR =

$$y_n + hf + \frac{h^2}{2} f f_y + \frac{h^3}{6}(f f_y^2 + f^2 f_{yy}) \frac{1}{24} h^4 (f^3 f_{yyy} + f f_y^3 + 4 f^2 f_y f_{yy}) + \ldots$$

Hence $ERROR = TAYLOR - \frac{UPPER}{LOWER}$

or, (TAYLOR x LOWER) - UPPER = (LOWER x ERROR).

EXTENDED RUNGE - KUTTA METHOD BASED ON HAM

In the development of methods for solving ordinary differential equations, it is not clear whether the arithmetic mean is always the best choice. Naturally RK formulae, based on arithmetic mean, are the most convenient and flexible to apply. But there is no guarantee that they would yield more accurate results for all type of problems. Hence, the use of harmonic means in the functional values instead of the usual arithmetic mean may result in better accuracy for a certain class of problems. It may be noted that the harmonic mean of two quantities x_1 and x_2 is given by

$$\frac{2 x_1 x_2}{x_1 + x_2}$$

In [11], it has been shown that the use of harmonic means in the functional values, instead of the usual arithmetic mean in the trapezoidal formula has also produced a formula with an accuracy of order -2.

i.e., $$x_{n+1} = x_n + h\left(\frac{2 f_n f_{n+1}}{f_n + f_{n+1}}\right) \qquad (21)$$

The local truncation error (LTE) for the eq. (21) is given by

$$LTE = \left(\frac{-\dddot{x}_n}{12} + \frac{(\ddot{x}_n)^2}{4\dot{x}_n}\right) h^3 + 0(h^4)$$

It is possible to establish a 4-stage non-linear RK formula based on harmonic mean (RKHM) in the form

$$x_{n+1} = x_n + \frac{h}{3}\sum_{i=1}^{3}\left(\frac{2 k_i k_{i+1}}{k_i + k_{i+1}}\right)$$

i.e.,

$$x_{n+1} = x_n + \frac{h}{3}\left(\frac{2 k_1 k_2}{k_1 + k_2} + \frac{2 k_2 k_3}{k_2 + k_3} + \frac{2 k_3 k_4}{k_3 + k_4}\right)$$

as a direct extension of eq. (20), where

$$k_1 = f(x_n)$$
$$k_2 = f(x_n + h a_1 k_1)$$
$$k_3 = f(x_n + h(a_2 k_1 + a_3 k_2))$$
$$k_4 = f(x_n + h(a_4 k_1 + a_5 k_2 + a_6 k_3)).$$

Applying the formula of RKAM, RKCeM and RKHaM discussed in 4.2 - 4.4, the discrete solutions of (20) have been obtained, taking the step-size as h = 0.01, for different values of α^2.

DISCUSSION

Solving eq.(20) by the Laplace – Transform, the analytic expressions for $C_1(t)$ and $C_2(t)$ are

$$C_1(t) = 1.6408\, e^{-32.1807\alpha^2 t} + 2.3592\, e^{-2.486\alpha^2 t}$$

$$C_2(t) = -\left[2.265\, e^{-32.1807\alpha^2 t} + 1.068\, e^{-2.486\alpha^2 t}\right] \qquad (21)$$

The exact solution of eqs. (1) which satisfies the initial and boundary conditions given by eqs. (2) and (3) is obtained as (refer Ritger and Rose [115]).

$$T(x,t) = 2\sum_{n=0}^{\infty}\frac{e^{-\lambda_n^2 \alpha^2 t}\sin(\lambda_n x)}{\lambda_n} \qquad (22)$$

where $\lambda_n = (2n+1)(\pi/2)$

The numerical values of T(x,t), with different values of α^2 = 0.5, 0.75, 1.0 and 2.0 based on the value of x = 0.5 and 0.1, have been obtained by the methods of Ritz-Laplace Transform, Ritz-RKAM, Ritz-RKCeM, Ritz-RKHaM and the Ritz-STWS, and are respectively shown in Tables 1 – 4, together with their corresponding exact solutions. Also, the discrete solutions obtained by the methods RKAM and RKHaM, for the values of $C_1(t)$ and $C_2(t)$ of the eqs. (10), coincide well with the solutions obtained by the Laplace Transform. The numerical values of T(x,t), with different values of α^2 = 0.5, 0.75, 1.0 and 2.0 based on the value of x = 0.5 and 0.1, have been obtained by these two methods and are in good agreement with the exact solution (20). The obtained absolute error using the methods of Ritz-Laplace Transform, Ritz-RKAM, Ritz-RKCeM, Ritz-RKHaM and Ritz-STWS are given in Tables 5 – 12. For a sample, an error graph for x = 1 at time t = 1.4 is shown in Figure 1.

CONCLUSIONS

As an outcome of this study, new methods have been proposed for the investigation of heat-flow problem. The novel features of the present numerical schemes are the adoption of the Rayleigh – Ritz technique for the elimination of spatial dependency in the heat flow equation, the STWS and RK techniques for solving the resulting system of first order linear equations in time, and the Galerkin method for determining the initial conditions.

It is observed that Ritz-Laplace Transform, Ritz-STWS, Ritz-RKAM, Ritz-RKCeM and Ritz-RKHaM yield similar results. Reviewing these methods, applied for the heat-flow problem, it is clearly noticeable that Ritz-STWS, Ritz-RK methods involve less number of computations and the complexity of these methods are very simple. It is also to be noted that from Figure 1, the analytical method of Laplace Transform stands first, in respect to accuracy. However, RKCeM is found to yield better results among the other RK methods and STWS technique.

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