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RESEARCH ARTICLE

APPLICATIONS OF TENSOR IN VARIOUS SCIENTIFIC AND MATHEMATICAL FIELDS

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ABSTRACT

The analysis of different physical systems and mathematical devices depends on the utilization of various types of algebraic quantities involved in the description of geometrical aspects of the phenomenon and states which occur. The most familiar tensors in physical importance are internal stress in a solid and viscous stress in fluid. The purpose of the present manuscript is to discuss a nice and lucid characterization of tensor and their basic features. Moreover, the manuscript is intended to serve the purpose of familiarity with recent developments in the tensor theory and its applications in various fields of sciences. We just reviewed and combined the results on applications of tensors, which we feel useful in the recent developments. The manuscript begins with tensor primer and consists of few applications in elasticity including illustrations of stress and deformation tensors of elastic bodies, electro-dynamics with Maxwell's tensor and finally includes brief notions of diffusion tensors used in strain Green's tensors applied in the study of seismology.

INTRODUCTION

Since tensors are essentially a particular category of functions depending on vector manifolds, we shall start the manuscript with a sophomore fundamental level. A little bit concept of linear algebra along with the familiar material about vectors, scalars, bases and linear operators etc. will be recapitulated, but eventually, we shall move on to slightly more sophisticated topics that are essential for going through the proposed ramifications of tensors in Mathematics as well as physics. Here are few definitions of tensors:

Definition (1.1): In terms of components: A tensor is a set of components associated with a specific co-ordinate frame of system that transform according to specific rules under a change of the co-ordinate frame of system.

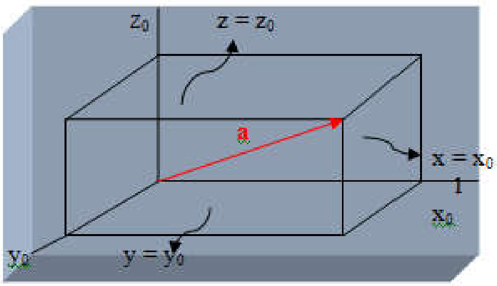

Figure (a): Component representation of a vector

Figure (a) shows the components representation of a vector. Co-ordinate frame i.e. surface (plane for Cartesian manifold) at the head of the vector tuple (x_0, y_0, z_0). Similarly, a three dimensional 2^{nd} rank tensor consists of 3^2 i.e. nine components arranged in a square matrix. A complete treatment of the transformation rules for the components of a tensor is realistically complex and involves selection (often arbitrary) of a group of transformation (linear, projective, conformal, affine etc.) functions and an associated component transformation which chooses the tensor quantities that preserve desired invariants. The detail evolutions of these issues are covered in [5].

Definition (1.2): In terms of linear map: A tensor is a *multi-linear map* on the other tensors. For instance, scalars are 0^{th}-order tensors that perform a transformation which scales (change the magnitude) of other scalars, vectors and 2^{nd} order tensors. Vectors can be seen as linear maps (through the inner product) which convert other vectors to scalars. 2^{nd} order tensors are bilinear maps which map one vector to another.

Definition (1.3): In terms of Geometric objects: A tensor is a geometric object that has certain invariant properties when viewed from different co-ordinate systems. A scalar, for example, is a single magnitude quantity that does not changed when the co-ordinate system is rotated. Real symmetric tensors of rank two can be viewed as a quadric surface such as an ellipse in 2D (Figure (b)) or ellipsoid in 3D. Under the rotation or translation of the co-ordinate system the direction and the magnitude of the quadric surface axes and the surface shape do not change.

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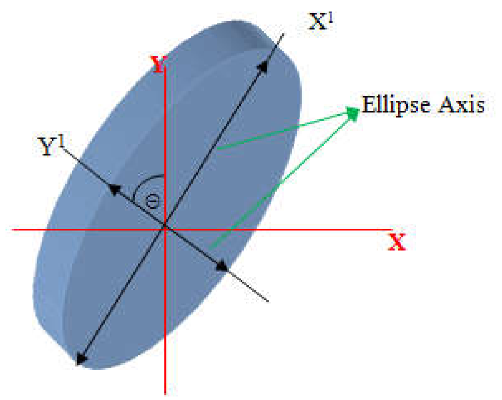

Figure (b): Geometry of 2nd rank real symmetric tensor

Definition (1.4): In terms of component freeness: We'll now define the modern component free definition a tensor which will follow the usual transformation laws.

A tensor of type (r, s) on a vector manifold V is a scalar valued (i.e. C-valued) function T on

$$\underbrace{V \times V \times \ldots \times V}_{r-times} \times \underbrace{V^* \times V^* \times \ldots \times V^*}_{s-times} \qquad (1.1)$$

which is linear in each argument, i.e.

$$T(v_1 + cw, v_2, \ldots, v_r, f_1, f_2, \ldots, f_s) = T(v_1, \ldots, v_r, f_1, \ldots, f_s) + cT(w, v_2, \ldots, f_1, \ldots, f_s) \qquad (1.2)$$

are similar for all arguments. This property is called *multi-linearity*. It is quite interesting to note that dual vectors are (1, 0) tensors and those vectors can be viewed as (0, 1) tensors as follows:

$$v(f) \equiv f(v) where\ v \in V, f \in V^*. \qquad (1.3)$$

Similarly, linear operators can be viewed as (1, 1) tensors as

$$A(v, f) \equiv f(Av). \qquad (1.4)$$

Applications of tensor in various fields

In this section, we shall outline some applications of tensor in various mathematical & scientific fields.

Applications in Elasticity (Stress tensor of elastic bodies) [2]: Surface stress is a specification of the force per unit area (force intensity) acting on the surface. The state of stress of an elastic medium is specified once we know the force working on an arbitrary element of area $d\sigma$ passing through an arbitrary point M of the medium which, while defining the stress should be considered in equilibrium. Let $\boldsymbol{r}$ be the radius vector of M and $\boldsymbol{n}$ be the unit normal to $d\sigma$. Then the force acting on $d\sigma$ equals to $\boldsymbol{p}d\sigma$, where the stress $\boldsymbol{p}$ is a function of $\boldsymbol{p}(\boldsymbol{r}, \boldsymbol{n})$ for the vectors $\boldsymbol{r}$ and $\boldsymbol{n}$ (Figure 2.1). Now, we shall show that the function $\boldsymbol{p}(\boldsymbol{r}, \boldsymbol{n})$ can be deduced from a certain second order tensor called *stress tensor*, which depends on $\boldsymbol{p}(\boldsymbol{r}, \boldsymbol{n})$, $\boldsymbol{r}$ but not on $\boldsymbol{n}$. To this end, we fabricate an elementary tetrahedron about the point M with its edges directed along the axes of a rectangular Cartesian co-ordinate system K (Fig. 2.2). Suppose $d\sigma_1, d\sigma_2\ and\ d\sigma_3$ represent the area of the faces perpendicular to the axes x_1, x_2 and x_3 and let $d\sigma_n$ denotes the area of the inclined face with unit exterior normal $\boldsymbol{n}$.

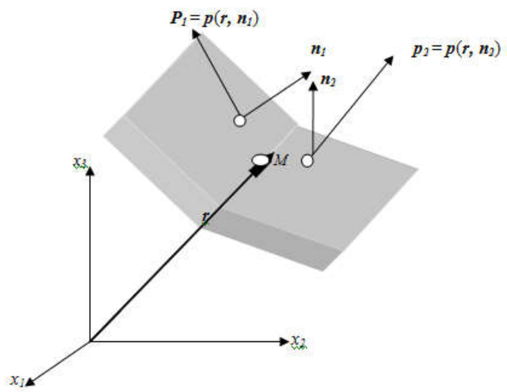

Fig. 2.1: The Stress acting on an element of area in an elastic medium depends on both the position and the orientation of the element

Furthermore, let $\boldsymbol{p}_{-1}\, d\sigma_1$, $\boldsymbol{p}_{-2}\, d\sigma_2$, $\boldsymbol{p}_{-3}\, d\sigma_3$ and $\boldsymbol{p}_{-n}\, d\sigma_n$ be the forces imposed by the rest of the medium on the areas $d\sigma_1, d\sigma_2, d\sigma_3\ and\ d\sigma_n$, respectively. Here the minus sign means that the stresses $\boldsymbol{p}_{-1}$, $\boldsymbol{p}_{-2}$ and $\boldsymbol{p}_{-3}$ are acting on the outside faces of the tetrahedron, whose exterior normal point in the directions opposite to those of the co-ordinate axes. By the law of action and reaction, the forces $\boldsymbol{p}_1\, d\sigma_1$, $\boldsymbol{p}_1\, d\sigma_2$ and $\boldsymbol{p}_3\, d\sigma_1$ acting on the inside faces of the tetrahedron are equal and opposite to those acting on the outside faces, and hence

$$\boldsymbol{p}_1 = -\boldsymbol{p}_{-1},\ \boldsymbol{p}_2 = -\boldsymbol{p}_{-2},\ \boldsymbol{p}_3 = -\boldsymbol{p}_{-3}.$$

Now, let $\mathbf{a}$ be the acceleration of the centre of mass of the tetrahedron and let $\mathbf{f}$ be the body force per unit mass. Then, by Newton's second law,

$$\begin{aligned}\mathbf{a}\, dm &= \mathbf{f}\, dm + \boldsymbol{p}_n d\sigma_n + \boldsymbol{p}_{-1} d\sigma_1 + \boldsymbol{p}_{-2} d\sigma_2 + \boldsymbol{p}_{-3} d\sigma_3 \\ &= \mathbf{f}\, dm + \boldsymbol{p}_n d\sigma_n - \boldsymbol{p}_1 d\sigma_1 - \boldsymbol{p}_2 d\sigma_2 - \boldsymbol{p}_3 d\sigma_3,\end{aligned}$$

where dm is the mass of the tetrahedron. In the limit, as the tetrahedron shrinks to a point M, we obtain that

$$\boldsymbol{p}_n d\sigma_n = \boldsymbol{p}_1 d\sigma_1 + \boldsymbol{p}_2 d\sigma_2 + \boldsymbol{p}_3 d\sigma_3 = \textstyle\sum_{i=1}^{3} \boldsymbol{p}_i d\sigma_i,$$

since the terms containing dm are proportional to the volume of the tetrahedron and hence are of a higher order of smallness compared to the terms containing elements of area. Since $d\sigma_i = d\sigma_n \cos(\mathbf{n}, i) = n_i d\sigma_i$, therefore the stress on an element of area with unit normal $\mathbf{n}$ is given by $\boldsymbol{p}_i = \sum_{i=1}^{3} \boldsymbol{p}_i n_i = \boldsymbol{p}_i n_i$.

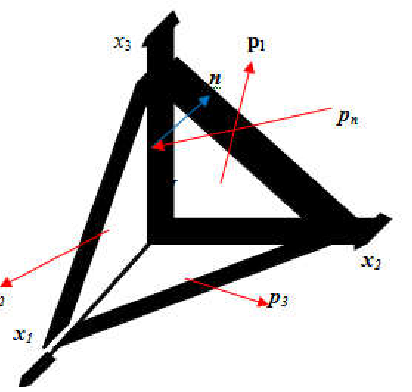

Fig. 2.2: Stresses on the faces on tetrahedron

Projecting $\boldsymbol{p}_n$ onto the axes of the system *K*, we have

$$\boldsymbol{p}_{nk} = \boldsymbol{p}_{ik}\boldsymbol{n}_i,$$

where $\boldsymbol{p}_{ik}(i = 1, 2, 3)$ is a set of nine normal $(i = k)$ and tangential $(i \neq k)$ stresses acting on three orthogonal elements of area at the point *M* (see Fig. 2.3). Although $\boldsymbol{p}_{ik}$ is independent of the orientation **n** of the area on which the stress acts. These nine quantities, which depend only on the point *M*, allow us to determine $\mathbf{p_n}$ for arbitrary **n**. Thus the physical quantity with components p_{ik} called *stress tensor* uniquely specifies the state of stress at every point of the elastic medium.

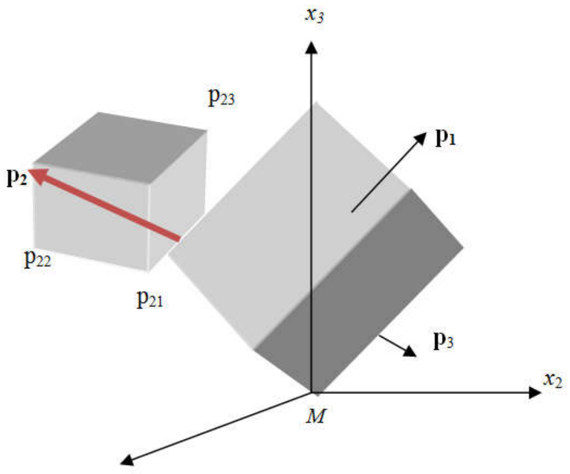

Fig. 2.3: The stress tensor as a set of three stress vectors $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ acting on three orthogonal elements of area. The projections of these vectors onto the co-ordinate axes give the nine components of the stress tensor

It only remains to verify the tensor character of p_{ik}. Since, the definition of p_{ik} involves no restriction on the normal **n**. We can assume without loss of generality that the *i*th axis of the new co-ordinate system K' is directed along **n**, so that

$\mathbf{N}=\mathbf{i}_i'$ (K and K' have orthonormal bases $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ and $\mathbf{i}_1{}'$ $\mathbf{i}_2{}'$ $\mathbf{i}_3{}'$, respectively). Then projecting **n** onto the *l*th axis of *K* produces

$$n_l = \mathbf{n}.i_l = i'_l.i_l = \alpha_{i'l},$$

where $\alpha_{i'l}$ is the cosine of the angle between the *i*th axis of K' and the *l*th axis of *K*, and hence

$$\boldsymbol{p}_n \equiv \boldsymbol{p_i}' = \boldsymbol{p}_l n_{l=}\alpha_{i'l}\boldsymbol{p}_l = \alpha_{i'l}\boldsymbol{i}_m p_{lm}.$$

Eventually, projecting $\boldsymbol{p_i}'$ onto the *k*th axis of K', we obtain

$$\boldsymbol{p_i}'\boldsymbol{i_k}' = \alpha_{i'l}(\boldsymbol{i}_m.\boldsymbol{i_k}')p_{lm} \quad \text{Or}$$
$$\boldsymbol{p_{ik}}' = \alpha_{i'l}\alpha_{k'm}p_{lm}.$$

Comparing the last equation with the standard equation of transformation of the second rank tensor, which is given by $A'_{ik} = \alpha_{i'l}\alpha_{k'm}A_{lm}$. Hence we observe that p_{ik} transforms like a second order tensor, as it is required.

Deformation tensor of elastic bodies [2]: Let A and B be the two vicinal points of an elastic body. Due to deformation in elastic body points A and B changes to A', B'. Let *A* and *B* have radius vectors **r** and **r+Δr**, while $A'$and B' have radius vectors **r+u(r)** and **r+Δr+u(r+Δr)**, as shown in figure (2.4). The vectors **u(r)** and **u(r+Δr)** describe the displacement of the points *A* and *B* as a result of the deformation. As shown in figure (2.4), the relative position of the points is given by $\Delta\boldsymbol{r}$ before the deformation and $\Delta\boldsymbol{r}' = A'B' = \Delta\boldsymbol{r} + \boldsymbol{u}(\boldsymbol{r} + \Delta\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{r})$ after the deformation. The change in magnitude of $\Delta\boldsymbol{r}$ can be found by calculating the quantity $(\Delta\boldsymbol{r}^i)^2 - (\Delta\boldsymbol{r})^2$. Suppose **u** is a adequately smooth function of position, with components $u_i = u_i(x_i, x_2, x_3)$.

Then $\Delta x_i' = \Delta x_i + u_i(x_1 + \Delta x_1, x_2 + \Delta x_2, x_3 + \Delta x_3) - u_i(x_1, x_2, x_3)$, Using Taylor's theorem and neglecting terms of the second order of smallness

Or $$\Delta x_i' = \Delta x_i + \frac{\partial u_i}{\partial x_k}\Delta x_k \tag{2.1}$$

Again, $\Delta x_i'\Delta x_i' = (\Delta r')^2$, $\Delta x_i \Delta x_i = (\Delta r)^2$, and squaring Equation (2.1), we get

$$(\Delta r')^2 - (\Delta r)^2 = 2\frac{\partial u_i}{\partial x_k}\Delta x_i \Delta x_k + \frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_l}\Delta x_k \Delta x_l$$
$$= \left[\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} + \frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_k}\right]\Delta x_i \Delta x_k = 2u_{ik}\Delta x_i \Delta x_k,$$

where

$$u_{ik} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} + \frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_k}\right] \tag{2.2}$$

Thus, the change in the distance between any two points of the elastic body is uniquely expressed by the quantity u_{ik}, which can be transformed to a new co-ordinate frame system K' to obtain the following:

$$u'_{ik} = \frac{1}{2}\left[\frac{\partial u'_i}{\partial x'_k} + \frac{\partial u'_k}{\partial x'_i} + \frac{\partial u'_l}{\partial x'_i}\frac{\partial u'_l}{\partial x'_k}\right].$$

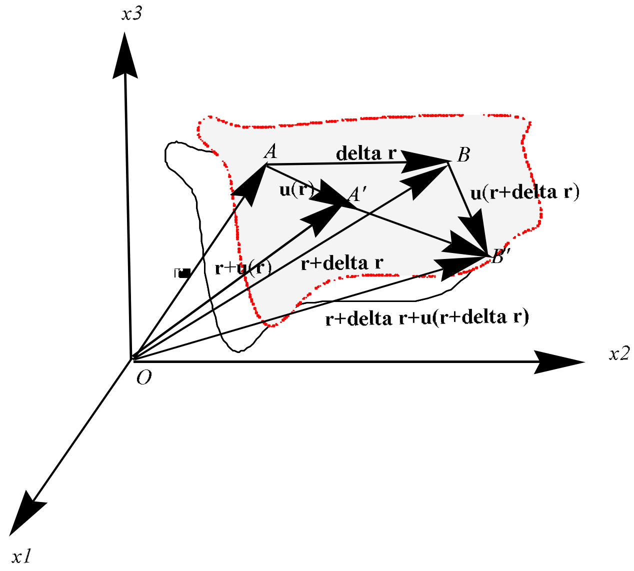

Fig. 2.4: Deformation of an elastic body

It follows from the transformation law $x_i = \alpha_{k'i}x_k' + x_{i0}$ explaining the transformation from K' back to the old co-ordinate frame system *K* and x_{i0} are the co-ordinates of the new origin in the old system, while

$$\alpha_{k'i} = \frac{\partial x_i}{\partial x'_k}. \tag{2.3}$$

The repeated use of transformation law, Eq. (2.3) and the chain rule for the partial derivation, we obtain

$$u'_{ik} = \frac{1}{2}[\frac{\partial}{\partial x_m}(\alpha_{i'm}u_m)\frac{\partial x_n}{\partial x'_k} + \frac{\partial}{\partial x_m}(\alpha_{k'n}u_n)\frac{\partial x_m}{\partial x'_i} + \frac{\partial}{\partial x_m}(\alpha_{l'r}u_r)\frac{\partial x_m}{\partial x'_i}\frac{\partial}{\partial x_n}(\alpha_{l's}u_s)\frac{\partial x_n}{\partial x'_k}]$$

$$= \frac{1}{2}(\alpha_{i'm}\frac{\partial u_m}{\partial x_n}\alpha_{k'n} + \alpha_{k'n}\frac{\partial u_n}{\partial x_m}\alpha_{i'm} + \alpha_{l'r}\frac{\partial u_r}{\partial x_m}\alpha_{i'm}\alpha_{l's}\frac{\partial u_s}{\partial x_n}\alpha_{k'n})$$

$$= \alpha_{i'm}\alpha_{k'n}\frac{1}{2}(\frac{\partial u_m}{\partial x_n} + \frac{\partial u_n}{\partial x_m} + \delta_{rs}\frac{\partial u_r}{\partial x_m}\frac{\partial u_s}{\partial x_n})$$

$$= \alpha_{i'm}\alpha_{k'n}\frac{1}{2}(\frac{\partial u_m}{\partial x_n} + \frac{\partial u_n}{\partial x_m} + \frac{\partial u_r}{\partial x_m}\frac{\partial u_s}{\partial x_n}),$$

i.e. $u'_{ik} = \alpha_{i'm}\alpha_{k'n}u_{mn}$

It follows that u_{ik} is a second rank tensor as evident from the transformation law of second rank tensor. In the linear theory of elasticity, the terms, such as $\frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_k}$ are dropped and due to this Eq. (2.2) will have the only term as $u_{ik} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right]$.

Applications of tensors in Electro-Dynamic[4](Maxwell Stress Tensor): While considering the conservation of total momentum (mechanical plus electromagnetic) in electrodynamics one comes upon the symmetric second rank *Maxwell stress tensor*, defined in (2, 0) form as

$$T_{(2,0)} = \boldsymbol{E} \otimes \boldsymbol{E} + \boldsymbol{B} \otimes \boldsymbol{B} - \frac{1}{2}(\boldsymbol{E}.\boldsymbol{E} + \boldsymbol{B}.\boldsymbol{B})g,$$

where the sign $\otimes$ is used for tensor product and ***E, B*** are the dual vector versions of the electric and magnetic fields respectively. *T* can be described as *T* (*v, w*) which produces the rate at which momentum in the *v*- direction flows in the *w*-direction. In terms of components, we can write the above expression as follows:

$$T_{ij} = E_iE_j + B_iB_j(\boldsymbol{E}.\boldsymbol{E} + \boldsymbol{B}.\boldsymbol{B})\delta_{ij},$$

which is the crucial expression often seen in classical electro-dynamics

The electro-magnetic field tensor: In relativistic electro-dynamics, the electric and magnetic field vectors are properly observed as components of a second rank anti-symmetric tensor *F*, the electro-magnetic field tensor which is defined as

$$F_{[2,0]} \equiv F^{ij} = \begin{Vmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{Vmatrix}$$

The Lorentz force law $\frac{dp^\mu}{d\tau} = qF^\mu_\vartheta v^\vartheta$, where $p = mv$ is the 4-momentum of a particle, v is its proper velocity and q its charge, can be re-written without components as $\frac{dp}{d\tau} = qF_{[1,1]}(v)$ which just evokes that the proper force on a particle is given by the action of the field tensor on the particle's proper velocity.

Strain Green's Tensors & Their Significance to Seismology [3]:

Fundamental Aspect: The approaches of Green's function are extensively used in the modeling of seismic wave forms. Seismic waveforms of finite frequency carry a great deal of information on earthquake sources and earth structure and hence the numerical waveform modeling tools provide the wave field solutions caused by earthquake sources that are either point or distributed moment tensors. They can also suggest the *Green's functions*, the wave fields from unit impulsive point force that are very useful in the seismic study. Green's functions are mainly very helpful in improving the computational efficiency when there is a need to calculate the waveforms for the same source. In calculating the waveforms from the source of earthquake, described by moment tensors, it is in fact the spatial gradients of the Green's tensors that are directly used. Strain Green's Tensors: In seismology, the strain Green's tensor is often used to describe the deformation of the earth medium caused by seismic wave-generated displacement field and is linearly related to the stress field by the constituted law. It is a second order symmetric tensor defined in terms of the spatial-gradient elements of the displacement vector:

$$\varepsilon(\boldsymbol{r}, t) = \frac{1}{2}[\{\nabla\boldsymbol{u}(\boldsymbol{r}, t)\} + \{\nabla\boldsymbol{u}(r, t)\}^T]. \qquad (2.4)$$

Similar to the definitions, we can form a third order tensor by using the spatial gradient elements of the second order Green's tensor:

$$\boldsymbol{H}(\boldsymbol{r}, t;\, \boldsymbol{r_s}) = \frac{1}{2}[\{\nabla\boldsymbol{G}(\boldsymbol{r}, t;\, \boldsymbol{r_s})\} + \{\nabla\boldsymbol{G}(\boldsymbol{r}, t;\, \boldsymbol{r_s})\}^T]. \qquad (2.5)$$

In Eq. (2.7), $\boldsymbol{r_s}$ is the source location of the Green's tensor, the spatial gradient operator acts on the field co-ordinate **r**, and the notation $\{.\}^T$ indicates the transposition of the first two indices of a third order tensor, that is $\{H_{inj}\}^T = H_{nji}$. In components form, Eq. (2.5) can be written as

$$H_{inj}(\boldsymbol{r}, t;\, \boldsymbol{r_s}) = \frac{1}{2}[\partial_i G_{jn}(\boldsymbol{r}, t;\, \boldsymbol{r_s}) + \partial_j G_{in}(\boldsymbol{r}, t;\, \boldsymbol{r_s})]. \qquad (2.6)$$

This third rank tensor represents the strain associated with the Green's tensors and we call it the *strain Green's tensor (SGT)*. The (*SGT*) is symmetric with respect to its first pair index and therefore it has only eighteen independent elements. In seismology, many expressions involve the spatial gradient of the Green's tensor instead of Green's tensor itself; as a result, the (*SGT*) defined in the equations (2.7) and (2.8) is a often more immediately used quantity and making it available can frequently improve the efficiency in numerical calculations. For instance, the displacement field from a point double-couple earthquake source can be expressed as [1]:

$$u_n(\boldsymbol{r}, t;\, \boldsymbol{r_s}) = \partial^S{}_i G_{nj}(\boldsymbol{r}, t;\, \boldsymbol{r_s})M_{ji}, \qquad (2.7)$$

where **M** is the moment tensor and the superscript *S* to the gradient operator indicates that it acts on the source co-ordinates. Taking the symmetry of the moment tensor into account and applying the reciprocity of the Green's tensor, equation (2.7) can be written as:

$$u_n(\boldsymbol{r},t;\,\boldsymbol{r_s}) = \frac{1}{2}[\partial^S{}_i G_{nj}(\boldsymbol{r_s},t;\,\boldsymbol{r}) + \partial^S{}_j G_{ni}(\boldsymbol{r_s},t;\,\boldsymbol{r})]M_{ji}. \quad (2.8)$$

Thus, we have

$$u_n(\boldsymbol{r},t;\,\boldsymbol{r_s}) = H_{ijn}(\boldsymbol{r_s},t;\,\boldsymbol{r})M_{ji} \text{ or } \boldsymbol{u}(\boldsymbol{r},t;\,\boldsymbol{r_s}) = \boldsymbol{M}\colon \boldsymbol{H}(\boldsymbol{r_s},t;\,\boldsymbol{r}). \quad (2.9)$$

Equation (2.9) provides an immediate linear relationship between the displacement and the moment tensor. Therefore, the elements of the (*SGT*) can be used in earthquake source-parameter inversions to obtain partial derivatives of the seismic data with respect to the moment tensor elements. Because of the (*SGT*) is immediately linked with the earthquake generated displacement field, the capability of efficiently providing the (*SGT*) can greatly improve the efficiency in the modeling of seismic data.

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