Available online at http://www.journalcra.com

International Journal of Current Research
Vol. 8, Issue, 03, pp. 28273-28283, March, 2016

INTERNATIONAL JOURNAL OF CURRENT RESEARCH

ISSN: 0975-833X

RESEARCH ARTICLE

EUCLIDEAN AND NAPOLEONIAN THEOREMS: THEIR DERIVATION FROM PYTHAGOREAN

***Cristopher John S. Rosero**

Department of Mathematics, Cebu Normal University, Cebu City, Cebu, Philippines

ARTICLE INFO

Article History:
Received 18th December, 2015
Received in revised form
24th January, 2016
Accepted 26th February, 2016
Published online 31st March, 2016

Key words:
Euclidean Theorem,
Napoleonian Theorem,
Pythagorean Theorem.

ABSTRACT

This research focuses mainly on the three most elegant statements of triangle geometry, namely, *Pythagorean*, *Euclidean*, and *Napoleonian* theorems. In classical geometry, *Pythagorean Theorem* states that if one is to construct squares to each of the sides of any right triangle, then the area of the square constructed at the hypotenuse is equal to the sum of the areas of the other two squares constructed on the other two sides, while *Euclidean Theorem* states that if one is to construct similar figures to each of the sides of any right triangle, then the area of the figure constructed at the hypotenuse is equal to the sum of the areas of the other two figures constructed on the other two sides. On the other hand, *Napoleonian Theorem* states that if one is to construct equilateral triangles on the sides of any triangle, the centers of those equilateral triangles themselves form an equilateral triangle. Here, the researcher investigates the three above-said theorems and in the process provides proof of *Euclidean Theorem* using the *Pythagorean Theorem*, then goes to prove *Napoleonian Theorem* using again the Pythagorean *Theorem*.

Copyright © 2016, Cristopher John S. Rosero. *This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.*

Citation: Cristopher John S. Rosero, 2016. "Euclidean and napoleonian theorems: Their derivation from pythagorean", *International Journal of Current Research,* 8, (03), 28273-28283.

INTRODUCTION

In classical geometry, one of the most celebrated and reputedly most proven theorems, is the Pythagorean Theorem, which states the relationship between and among the squares constructed on the sides of an arbitrary right triangle. A little over two hundred years thereafter, Euclid generalized the Pythagorean Theorem, stating in effect that the relationship that Pythagoras observed among and in between the squares constructed on the sides of any given right triangle is also true if similar figures are constructed on said sides, instead of squares. Another two thousand years or so, the twice-emperor of France, Napoleon Bonaparte, rediscovered a theorem that bears his name. The said theorem describes the nature and property of the equilateral triangles that are constructed in any given arbitrary triangle. There is reason to believe that the theorems of Pythagoras, Euclid and Napoleon are mathematically interconnected. In his book, Geometry by Discovery, David Gay noted that "...geometry is more than a series of events. There's additional spice when there are links between them and surprises occur when one come upon the links..."(Gay, 1998). The interconnection among and in between the theorems of Pythagoras, Euclid and Napoleon is the main underlying factor that innervates the researcher to undergo this study.

Theoretical Background

Rectilinear and Similar Figures

In classical geometry, a *polygon* consists of three or more coplanar segments; the segments, called the *sides*, intersect only at endpoints; each endpoint, called the *vertex*, belongs to exactly two segments, and no two segments with a common endpoint are collinear. A *rectilinear figure*, or simply referred to as a *figure*, is that which is contained by any boundary or boundaries. All throughout this study, however, the terms polygon, geometric figure, and figure are used interchangeably. A polygon with all its sides equal is *equilateral*, whereas, that in which all its angles are equal is *equiangular*. Two or more figures are said to be *similar* if their angles are correspondingly equal and the sides that contain each of the corresponding equal angles are correspondingly proportional.

Corresponding author: Cristopher John S. Rosero,
Department of Mathematics, Cebu Normal University, Cebu City, Cebu, Philippines.

Prominent all throughout this research study is the *triangle*, which is a three-sided closed figure. The angle thus formed by any two adjacent sides of a triangle is said to be an *interior angle.* A striking difference between *classical* and *non-classical geometry* is the sum of the said three interior angles of a given triangle. In *classical geometry*, the sum of the interior angles of any triangle is equal to two right angles, whereas in the *non-classical geometry*, the sum of the interior angles of any triangle, though it may equal to two right angles, yet it can be less than or greater than two right angles.

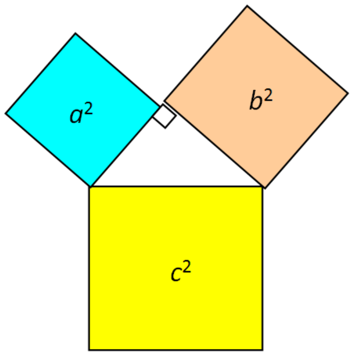

Figure 1. A Right-Angled Triangle with a Square

Pythagorean Theorem

The *Pythagorean Theorem*, named after the Greek mathematician-philosopher Pythagoras of Samos, (circa 569 BC – circa 475 BC), states that for a right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides (see Figure 1). It is worthy to note here that to Pythagoras and to the Greeks in general during that time, the square on the hypotenuse is not to be thought of as a number multiplied by itself, but rather as a geometrical square constructed on the side. To say that the square on the hypotenuse of a right-angled triangle equals the sum of the squares on the other two sides means that the two squares could be cut up and reassembled to form a square identical to the third square, as shown in Figure 2 below:

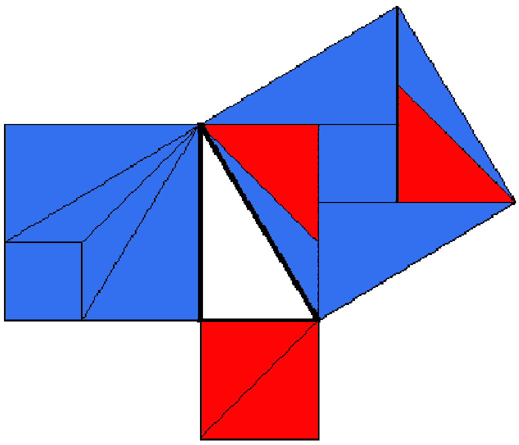

Figure 2. A Diagram of Pythagoras' Interpretation of His Theorem

Whether Pythagoras or someone else from the society of which he led, indeed was the first to discover its proof cannot be claimed with any degree of credibility. As what the mathematics historian William Dunham aptly describes it, "…there are those who doubt as to how Pythagoras did it, others who doubt that Pythagoras ever did it, and still others who doubt that Pythagoras even existed." (Dunham, 1994:91). The society that Pythagoras led, which is half -religious and half-scientific, followed a code of secrecy, and thus, unlike many later Greek mathematicians where some of the books which they wrote survive to this day, nothing of Pythagoras' writings, if there is any, survives.

Euclidean Theorem

Euclid of Alexandria, (circa 300 BC), was born about two hundred years after the death of Pythagoras, and is generally acknowledged as the Father of Geometry. He published a great number of works on a variety of topics, but he is most remembered for the thirteen-volume textbook, entitled Elements. Here, Euclid presents, in an eminently logical way, all of the elementary Greek geometrical knowledge of his day. This includes the theorems and constructions of plane geometry and solid geometry, along with the theory of proportions, incommensurables and commensurable, number theory, and a type of geometrical algebra (Boyer, 1989:119). The Greek mathematics historian Proclus, Euclid's foremost biographer, emphasized that a few original theorems in the Elements are directly attributed to Euclid, among others, is Proposition VI.31, which is a generalization of

Pythagorean Theorem, and which states: In right-angled triangles, the area of the figure on the side subtending the right angle is equal to the sum of the areas of the similar and similarly described figures on the sides containing the right angle. Note that in this proposition, henceforth referred to as *Euclidean Theorem*, the figures surrounding the middle triangle may be of any shape as long as they are all similar. In particular, in Figure 3, if triangle ABC is a right triangle, then the area of triangle ABC' is equal to the sum of the areas of triangles $AB'C$ and $A'BC$.

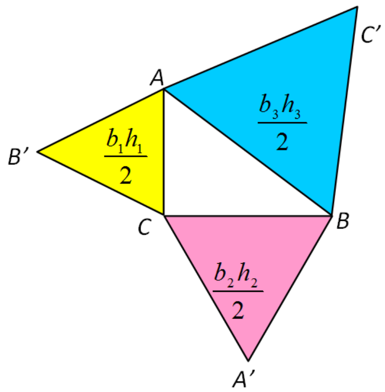

Figure 3. Right ΔABC with ΔABC', ΔAB'C, and ΔA'BC Constructed from each of its Side

Napoleonian Theorem

Napoleon Bonaparte (1769-1821) who came to power in 1799 and ruled France for 15 years. Known for his military endeavours and sheer personality, he made not only war strategies, but also a rediscovery of a mathematical, in particular, a geometric theorem. Generally known today as *Napoleon's Theorem*, it states that if one is to construct equilateral triangles on the sides of any triangle, the centers of those equilateral triangles themselves, once connected by straight lines, form an equilateral triangle, (see Figure 4).

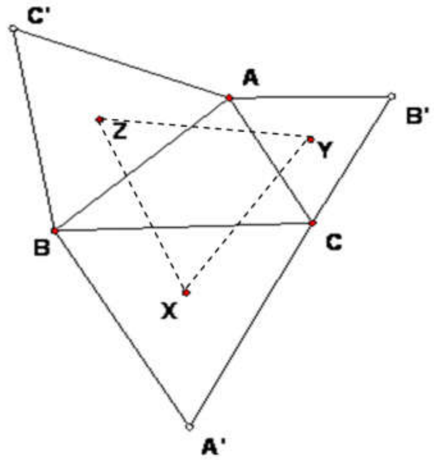

Figure 4. Equilateral Triangle XYZ Formed by Joining the Centers of Equilateral Triangles BCA', CAB'

In the figure, ABC is the starting triangle with an equilateral triangle built on each side: BCA' (on side $\overline{BC}$), CAB' (on side $\overline{AC}$)', and ABC' (on side $\overline{AB}$). The centers of the equilateral triangles are points X, Y, Z, respectively. If one is to join these three points by straight lines as shown in Figure 4, an equilateral triangle is formed (ΔXYZ). Talking about triangles, it is said that when two triangles are similar, the reduced ratio of any two corresponding sides is called the scale factor of the similar triangles. In Figure 5, ΔABC is similar to ΔDEF.

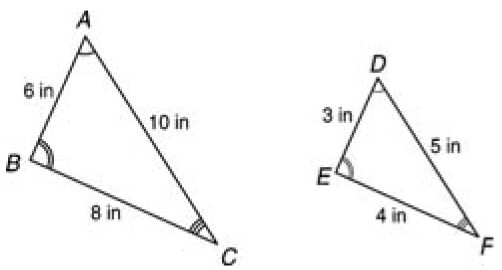

Figure 5. Similar Triangles ABC and DEF with a Scale Factor of 2:1

The ratios of corresponding sides are $\frac{6}{3}$, $\frac{8}{4}$, and $\frac{10}{5}$, respectively, which all reduce to $\frac{2}{1}$. It is then said that the scale factor of these two similar triangles is 2:1, (read as "2 is to 1"). The theorem stated without proof below states an important result in geometry;

Theorem 2.1.

If two similar triangles have a scale factor of $\boldsymbol{a}:\boldsymbol{b}$, then the ratio of their areas is $\boldsymbol{a}^2:\boldsymbol{b}^2$.
A much more important result that generalizes the ratio between the areas of similar polygons is stated without proof below.

Theorem 2.2.

The ratio of the areas of two similar polygons is the square of the ratio of the lengths of any two corresponding sides.

Euclidean Theorem and Napoleonian Theorem: Their Derivation from Pythagorean Theorem

Euclidean Theorem and its Proof

Heath's translation of Euclid's Book VI.31 states that in right-angled triangles the figure on the side subtending the right angle is equal to the sum of the areas of the similar and similarly described figures on the sides containing the right angle, of which, without loss of generality, can be stated as follows:

Theorem 3.1

If similar polygons are constructed on the sides of a right triangle, then the area of the polygon constructed at the hypotenuse is equal to the sum of the areas of that constructed at the legs.
Observe that by virtue of Theorem 2.2, it shall suffice to prove the Euclidean Theorem in the case of arbitrary similar triangles to prove that it is also true in the case of arbitrary similar polygons.

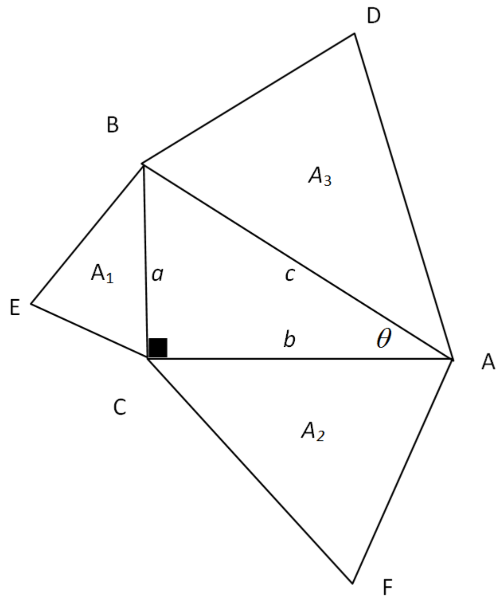

Figure 6. Similar Triangles ABD, BEC, and CAF Constructed at the Sides of Right Triangle ACB

Consider the right triangle, ΔACB with sides a, b, and hypotenuse c as shown in Figure 6. On each of its sides a and b are constructed two similar triangles which are ΔBCE and ΔACF respectively. Likewise, on its hypotenuse c, is a third triangle, ΔABD constructed in such a way that it is also similar to the two other triangles (ΔBCE and ΔACF). Letting the areas of these triangles be: A_1 for ΔBCE; A_2 for ΔACF and A_3 for ΔABD, one has to show that $A_3 = A_1 + A_2$.
Referring to the same figure (Figure 6) and applying Theorem 2.2, one has the following results:

$$\frac{A_1}{A_2} = \frac{a^2}{b^2} \; ; \; \frac{A_2}{A_3} = \frac{b^2}{c^2} \; ; \; and \; \frac{A_3}{A_1} = \frac{c^2}{a^2}$$

However, from the same figure, it is also clear that $\frac{a^2}{b^2} = \left(\frac{a}{b}\right)^2 = (\tan\theta)^2$ which can be written as $\tan^2\theta$. Since $\frac{A_1}{A_2} = \frac{a^2}{b^2}$ and $\frac{a^2}{b^2} = \tan^2\theta$, then $\frac{A_1}{A_2} = \tan^2\theta$.

Similarly for $\frac{A_2}{A_3}$ and $\frac{A_3}{A_1}$, the following are the results:

$$\frac{A_2}{A_3} = \frac{b^2}{c^2} = \left(\frac{b}{c}\right)^2 = (\cos\theta)^2 = \cos^2\theta$$

and,

$$\frac{A_3}{A_1} = \frac{c^2}{a^2} = \left(\frac{c}{a}\right)^2 = (\csc\theta)^2 = \csc^2\theta$$

Thus, the ratios of the areas of the three similar triangles in terms of the sides of the right triangle ACB and θ are shown as follows:

i. $\frac{A_1}{A_2} = \frac{a^2}{b^2} = \tan^2\theta \Rightarrow A_1 = A_2\tan^2\theta$

ii. $\frac{A_2}{A_3} = \frac{b^2}{c^2} = \cos^2\theta \Rightarrow A_2 = A_3\cos^2\theta$

iii. $\frac{A_3}{A_1} = \frac{c^2}{a^2} = \csc^2\theta \Rightarrow A_3 = A_1\csc^2\theta$

From equation iii;

$A_3 = A_1\csc^2\theta$

$= A_2\tan^2\theta\csc^2\theta$ (substituting A_1 by $A_2\tan^2\theta$ from equation i)

$= A_2\left(\frac{\sin^2\theta}{\cos^2\theta}\right)\left(\frac{1}{\sin^2\theta}\right)$ (since $\tan^2\theta = \left(\frac{\sin^2\theta}{\cos^2\theta}\right)$ and $\csc^2\theta = \left(\frac{1}{\sin^2\theta}\right)$)

$= A_2\left(\frac{1}{\cos^2\theta}\right)$ (algebraic manipulation)

$= A_2\sec^2\theta$ ($\left(\frac{1}{\cos^2\theta}\right) = \sec^2\theta$)

$= A_2(\tan^2\theta + 1)$($\sec^2\theta = (\tan^2\theta + 1)$)

$= A_2\tan^2\theta + A_2$ (algebraic manipulation)

Since, $A_2\tan^2\theta = A_1$ (from equation i), then

$A_2\tan^2\theta + A_2 = A_1 + A_2$.

Therefore

$A_3 = A_1 + A_2$

which proves the Euclidean Theorem.

Euclidean Theorem Proof Using Pythagorean Theorem

Classical geometry offers an area formula that deals with the triangle of which all the angle measurements are known but only one side measurement is known, appropriately illustrated below.

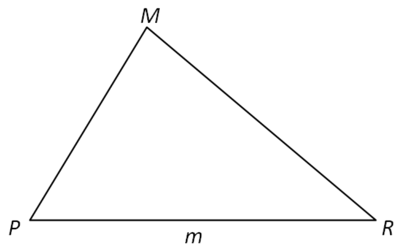

Figure 7. ΔPMR with a known side m and three known angles: P,M, and R

Based on classical geometry and from Figure 7, if PMR is a triangle of which angles P, M, and R are known, and side m is also known, then its area is given by the equation:

Area of $\Delta\ PMR = \dfrac{m^2 \sin P \sin R}{2 \sin M}$

With this area formula at hand, Euclidean Theorem can also be proven using the Pythagorean Theorem. Observe that by virtue of Theorem 3.1, it shall suffice to prove the Euclidean Theorem in the case of arbitrary similar triangles to prove that it is also true in the case of arbitrary similar polygons, thus one considers similar triangles A_1, A_2, and A_3, with corresponding angles, measuring, α, β, and γ respectively, constructed at the sides of a given right triangle with A_1 and A_2 constructed at the legs a, and b, respectively, and triangle A_3 constructed at the hypotenuse c, (see the Figure 8).

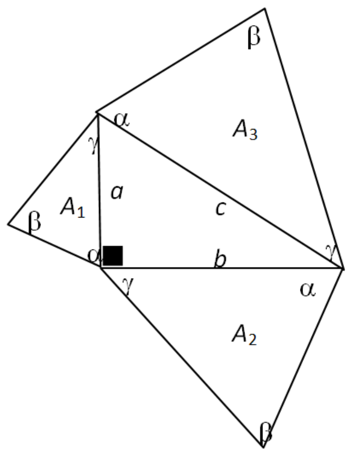

Figure 8. Similar Triangles Constructed at the Sides of a Right Triangle

As observed from Figure 8, and by applying the classical geometry area formula:

Area of $\Delta A_3 = \dfrac{c^2 \sin\alpha \sin\gamma}{2\sin\beta}$

$= \dfrac{(a^2 + b^2)\sin\alpha \sin\gamma}{2\sin\beta}$ [Pythagorean Theorem]

$= \dfrac{a^2 \sin\alpha \sin\gamma + b^2 \sin\alpha \sin\gamma}{2\sin\beta}$ [Algebraic Manipulation]

$$= \frac{a^2 \sin\alpha \sin\gamma}{2\sin\beta} + \frac{b^2 \sin\alpha \sin\gamma}{2\sin\beta} \text{ [Algebraic Manipulation]}$$

Area of ΔA_3= Area of ΔA_1 + Area of ΔA_2, [by the classical geometry area formula].

This shows that the area of triangle A_3 constructed on the hypotenuse is equal to the sum of the areas of the other triangles constructed on leg a (ΔA_1), and b (ΔA_2) respectively, hence proving the Pythagorean Theorem.

A closer observation in the derivation shows that
Area of ΔA_3 = Area of ΔA_1 + Area of ΔA_2

$$\Rightarrow \frac{c^2 \sin\alpha \sin\gamma}{2\sin\beta} = \frac{a^2 \sin\alpha \sin\gamma}{2\sin\beta} + \frac{b^2 \sin\alpha \sin\gamma}{2\sin\beta}$$

$$\Rightarrow c^2\left(\frac{\sin\alpha \sin\gamma}{2\sin\beta}\right) = a^2\left(\frac{\sin\alpha \sin\gamma}{2\sin\beta}\right) + b^2\left(\frac{\sin\alpha \sin\gamma}{2\sin\beta}\right)$$

which can be rewritten as : $\left(\frac{\sin\alpha \sin\gamma}{2\sin\beta}\right)c^2 = \left(\frac{\sin\alpha \sin\gamma}{2\sin\beta}\right)a^2 + \left(\frac{\sin\alpha \sin\gamma}{2\sin\beta}\right)b^2$,

The last equality shows that the area of the similar triangles can be expressed as the product of a constant and the square of the corresponding side on which the similar triangle is constructed. Letting this constant be s then;

Area = s * (*corresponding side*)^2, where

$s = \left(\frac{\sin\alpha \sin\gamma}{2\sin\beta}\right)$, and

corresponding side refers to the sides a, b, and c, respectively.

Let the formula Area = s * (*corresponding side*)^2 be called the scaling formula for the areas of similar triangles constructed on the sides of a right triangle.

Beyond the Pythagorean Theorem: The Law of Cosines

Consider an arbitrary ΔABC with its base at side b, and height, h, such that an angle C of known measurement is the intersection of two sides, a and b, of known length (see Figure 9).

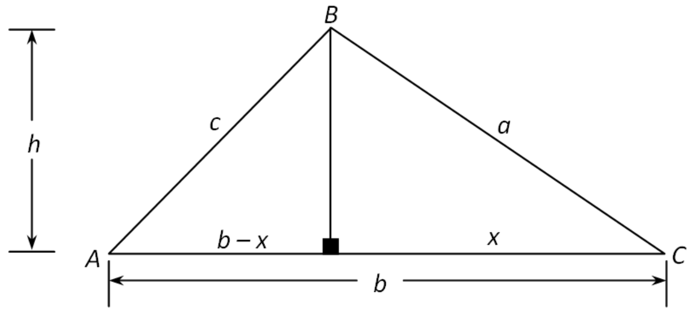

Figure 9. Triangle ABC with two sides a and b and the included angle measurements known

From Figure 9, the mathematical relationship between the sides a, b, and c are as follows:

(a) $h^2 = c^2 - (b-x)^2$ Pythagorean Theorem

(b) $h^2 = a^2 - x^2$ Pythagorean Theorem

(c) $c^2 - (b-x)^2 = a^2 - x^2$ Equating (a) and (b)

(d) $c^2 = a^2 + b^2 - 2bx$ Algebraic Manipulation

(e) $c^2 = a^2 + b^2 - 2ab\cos C$ $\cos C = \dfrac{x}{a} \Rightarrow x = a\cos C$

The last line, $c^2 = a^2 + b^2 - 2ab\cos C$, is the famous law of cosines and is actually the generalized Pythagorean Theorem in disguise.

Napoleonian Theorem and Its Proof

Napoleon's Theorem almost exclusively deals with equilateral triangles and their respective centers, and hence, the mathematical relationship between a given equilateral triangle and its center must first be investigated. The area of a given triangle with base *b* and height *h* is found by the formula, Area $= \frac{b \cdot h}{2}$. In the case of an equilateral triangle, it becomes, $\frac{s \cdot h}{2}$, where s is the length of the equilateral triangle's side. Another formula to find the area of an equilateral triangle is Area $= s^2 \cdot \frac{\sqrt{3}}{4}$, where, *s* is the equilateral triangle's side length. The height of an equilateral triangle in terms of its side length *s* is found by equating the two area formulas as follows:

$$\frac{s \cdot h}{2} = s^2 \cdot \frac{\sqrt{3}}{4} \Rightarrow h = s \cdot \frac{\sqrt{3}}{2}$$

One important property of an equilateral triangle is that the height *h* passes through the triangle's center, which is located at a length $\frac{2}{3}$ of the way from the top vertex down the *h* line, (see Figure 10).

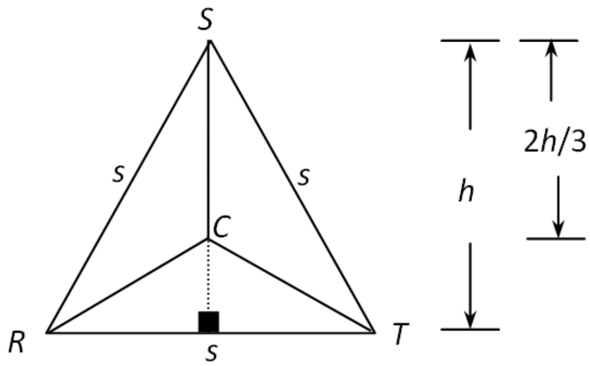

Figure 10. An Equilateral Triangle with center C

In terms of *s*, the length *d* from any of the vertices of an equilateral triangle to its center is:

$$d = \frac{2h}{3}$$

$$= \frac{2}{3} \cdot h$$

$$= \frac{2}{3} \cdot s \cdot \frac{\sqrt{3}}{2}$$

$$= \frac{s}{\sqrt{3}}$$

This result, in conjunction with the law of cosines, is of utmost importance in the derivation of Napoleon's Theorem, which is formally stated as follows:

Theorem 3.2.

If one constructs equilateral triangles on the sides of any triangle, the centers of those equilateral triangles themselves when connected by lines form an equilateral triangle.

Proof: Consider triangle ABC, with angular measurements $\alpha, \beta,$ and $\gamma,$ respectively. Constructing equilateral triangles at its sides, with centers at points, G, H, and I, respectively, a diagram shown in Figure 11 is obtained.

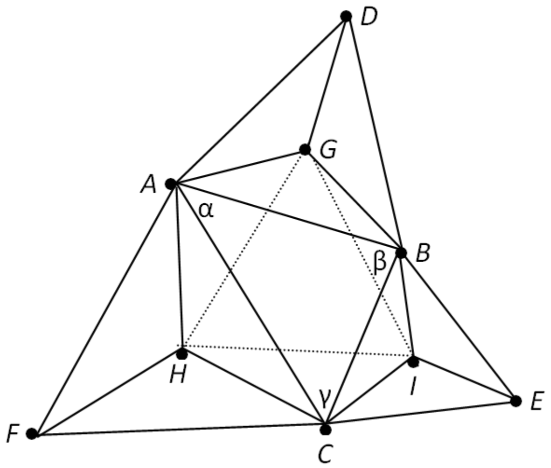

Figure 11. Equilateral Triangles ABD, ACF, and BCE, With centers, G, H, and I, respectively, constructed on the sides of Triangle ABC

Observe that $\angle HAG$, is the sum of angles HAC, α, and BAG, and since $\angle HAC = \angle BAG = 30°$, then $\angle HAG = \alpha + 60°$. The same observation shows that $\angle GBI = \beta + 60°$ and $\angle HCI = \gamma + 60°$, thus:

$$(GH)^2 = (AH)^2 + (AG)^2 - 2(AH)(AG)\cos(\alpha + 60°)$$

$$= \left(\frac{AC}{\sqrt{3}}\right)^2 + \left(\frac{AB}{\sqrt{3}}\right)^2 - 2\left(\frac{AC}{\sqrt{3}}\right)\left(\frac{AB}{\sqrt{3}}\right)\cos(\alpha + 60°)$$

$$= \frac{(AC)^2 + (AB)^2 - 2(AC)(AB)\cos(\alpha + 60°)}{3}$$

$$(GI)^2 = (BG)^2 + (BI)^2 - 2(BG)(BI)\cos(\beta + 60°)$$

$$= \left(\frac{AB}{\sqrt{3}}\right)^2 + \left(\frac{BC}{\sqrt{3}}\right)^2 - 2\left(\frac{AB}{\sqrt{3}}\right)\left(\frac{BC}{\sqrt{3}}\right)\cos(\beta + 60°)$$

$$= \frac{(AB)^2 + (BC)^2 - 2(AB)(BC)\cos(\beta + 60°)}{3}$$

$$(HI)^2 = (CI)^2 + (CH)^2 - 2(CI)(CH)\cos(\gamma + 60°)$$

$$= \left(\frac{BC}{\sqrt{3}}\right)^2 + \left(\frac{AC}{\sqrt{3}}\right)^2 - 2\left(\frac{BC}{\sqrt{3}}\right)\left(\frac{AC}{\sqrt{3}}\right)\cos(\gamma + 60°)$$

$$= \frac{(BC)^2 + (AC)^2 - 2(BC)(AC)\cos(\gamma + 60°)}{3}$$

Now, consider the original triangle *ABC* and the equilateral triangle *BCE* attached to it via side *BC*, (Figure 12).

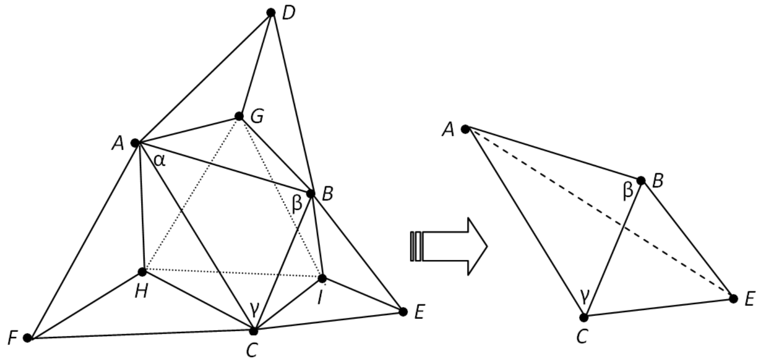

Figure 12. Triangle ABC and Equilateral Triangle BCE Attached to it via side BC

If one is to connect points *A* and *E*, and compute for the length of *AE* then:

From $\angle ABE$'s perspective,

$$(AE)^2 = (AB)^2 + (BE)^2 - 2(AB)(BE)\cos(\beta + 60°)$$

$$= (AB)^2 + (BC)^2 - 2(AB)(BC)\cos(\beta + 60°);$$

$$= 3 \cdot \frac{(AB)^2 + (BC)^2 - 2(AB)(BC)\cos(\beta + 60°)}{3}$$

$$= 3 \cdot (GI)^2$$

From $\angle ACE$'s perspective,

$$(AE)^2 = (AC)^2 + (CE)^2 - 2(AC)(CE)\cos(\gamma + 60°)$$

$$= (AC)^2 + (BC)^2 - 2(AC)(BC)\cos(\gamma + 60°);$$

$$= 3 \cdot \frac{(AC)^2 + (BC)^2 - 2(AC)(BC)\cos(\gamma + 60°)}{3}$$

$$= 3 \cdot (HI)^2$$

Since classical geometry allows only positive lengths for the sides, then without loss of generality,

$3 \cdot (GI)^2 = 3 \cdot (HI)^2 \Rightarrow GI = HI$.

Next, consider the original triangle *ABC* and the equilateral triangle *ACF* attached to it via side *AC*, (Figure 13).

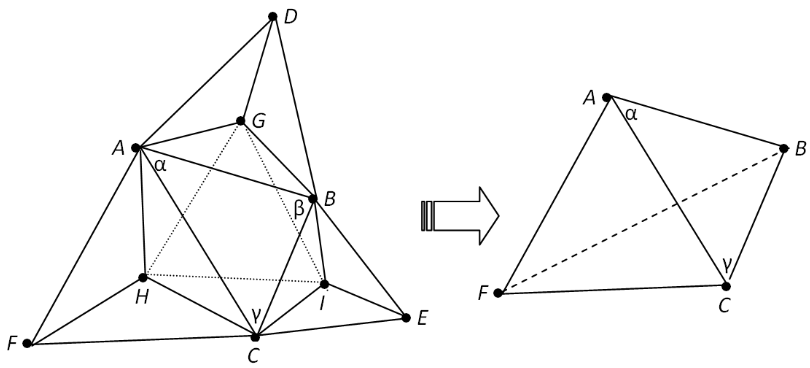

Figure 13. Triangle ABC and Equilateral Triangle AFC Attached to it via side AC

If one is to connect points B and F, and compute for the length of BF then:

From $\angle FAB$'s perspective,

$$(BF)^2 = (AF)^2 + (AB)^2 - 2(AF)(AB)\cos(\alpha + 60°)$$

$$= (AC)^2 + (AB)^2 - 2(AC)(AB)\cos(\alpha + 60°)$$

$$= 3 \cdot \frac{(AC)^2 + (AB)^2 - 2(AC)(AB)\cos(\alpha + 60°)}{3}$$

$$= 3 \cdot (GH)^2$$

From $\angle BCF$'s perspective,

$$(BF)^2 = (BC)^2 + (CF)^2 - 2(BC)(CF)\cos(\gamma + 60°)$$

$$= (BC)^2 + (AC)^2 - 2(BC)(AC)\cos(\gamma + 60°)$$

$$= 3 \cdot \frac{(BC)^2 + (AC)^2 - 2(BC)(AC)\cos(\gamma + 60°)}{3}$$

$$= 3 \cdot (HI)^2$$

This goes to show that $GH = HI$, and since $GI = HI$, then $GH = GI$, thus proving that ΔGHI is an equilateral triangle.

REFERENCES

Boyer, Carl, B., and Uta, C. Merzbach. A History of Mathematics. 2nd ed. New York: John Wiley Sons, Inc., 1989.
Dunham, William. The Mathematical Universe. New York: John Wiley Sons, Inc., 1993.
Gay, David. Geometry By Discovery. New York: John Wiley Sons, Inc.,1998.
Hooper, Alfred. Makers of Mathematics. New York: Random House, Inc., 1948
An Introduction to the Works of Euclid with an Emphasis on the Elements.
http://www.obkb.com/dcljr/euclid.html
Euclid, Elements (ed. Thomas L. Heath).
http://www.perseus.tufts.edu/cgi-bin/ptext?lookup=Euc.+toc

****