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RESEARCH ARTICLE

NUMERICAL AND ANALYTIC EXISTENCE OF PROPOSAL PROBLEM FORMULATION OVER FUZZY INTERVAL

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ABSTRACT

In this paper, we proved the convergence of the solution for the nonlinear fuzzy volterra integral classes equation over fuzzy interval with high computational and complexity to find the solution in analytical method, so we describable this solution by using Homotopy perturbation method, by using Banach fixed point theory for existence and uniqueness. That with explained numerical examples. Finally using the MAPLE program to solve our problem.

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1. INTRODUCTION

In the last two decades with the rapid development of nonlinear science, there has appeared ever- increasing interest of physicists and engineers in the analytical techniques for nonlinear problems. It is well known, that perturbation methods provide the most versatile tools available in nonlinear analysis of engineering problems (Nguyen, 1978). The perturbation methods, like other nonlinear analytical techniques, have their own limitations. At first, almost all perturbation methods are based on the assumption that a small parameter must exist in the equation. This so-called small parameter assumption greatly restricts applications of perturbation techniques. As is well known, an overwhelming majority of nonlinear problems have no small parameters at all. Secondly, the determination of small parameters seems to be a special art requiring special techniques. An appropriate choice of small parameters leads to the ideal results, but an unsuitable choice may create serious problems. Furthermore, the approximate solutions solved by perturbation methods are valid, in most cases, only for the small values of the parameters. It is obvious that all these limitations come from the small parameter assumption. These facts have motivated to suggest alternate techniques, such as variational iteration (He, 2000; Hillermeier, 2001), decomposition (Kaleva, 1987), expunction, variation of parameters (Liao, 1997; Nayef, 1985) and iterative (Nguyen, 1978; Puri and Ralescu, 1983). In order to overcome these drawbacks, combining the standard homotopy and perturbation method, which is called the homotopy perturbation, modifies the homotopy method. Many problems in natural and engineering sciences are modeled by partial differential equations (PDEs). These equations arise in a number of scientific models such as the propagation of shallow water waves, long wave and chemical reaction-diffusion models (Abbasbandy *et al*., 2007; Wu,1999). A substantial amount of work has been invested for solving such models. Several techniques including the method of characteristic, Riemann invariants, combination of waveform relaxation and multi-grid, (Wu and Ma, 1990; Rajab *et al*., 2013; SushilaRathora *et al*., 2012) periodic multi-grid wave form, variational iteration, homotopy perturbation and Adomian's decomposition (Rajab *et al*., 2013; SushilaRathora *et al*., 2012; Abbasbandy and Jafarian, 2006) have been used for the solutions of such problems. Most of these techniques en- counter the inbuilt deficiencies and involve huge computational work. He (Klir *et al*., 1997; Park *et al*., 1995) developed the homotopy perturbation method for solving linear, nonlinear, initial and boundary value problems by merging two techniques, the standard homotopy and the perturbation technique. The homotopy

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perturbation method was formulated by taking the full advantage of the standard homotopy and perturbation methods and has been applied to a wide class of functional equations. The basic motivation of the present paper is the implementation ofhis reliable technique for solving PDEs. In particular the proposed homotopy perturbation method (HPM) is tested on Helmholtz, Fisher's, Boussinesq, singular fourth-order partial differential equations, systems of partial differential equations and higher-dimensional initial boundary value problems. The proposed iterative scheme finds the solution without any discretization, linearization or restrictive assumptions and is free from round off errors. The HPM gives the solution in the form of a convergent series with easily computable components. Unlike the method of separation of variables which requires both initial and boundary conditions, the HPM gives the solution by using the initial conditions only. The fact that the proposed HPM solves nonlinear problems without using Adomian's polynomials can be considered as a clear advantage of this technique over the decomposition method.

2.Basic concepts

Basic definitions of fuzzy number are given in (1,2,10,15,17,20) as follows:

Definition 2.1.Fuzzy number.A fuzzy number is a map $u: R \to [a, b]$, which satisfying

(1)u is upper semi- continuous function,
(2)u(x) = 0 outside some interval $[a, d]$
(3)There are real numbersb,c such $a \le b \le c \le d$
i)u(x)is a monotonic increasing function on $[a, b]$
ii)u(x) is a monotonic decreasing function on $[c, d]$
iii)u(x) = 1 for all $x \in [b, c]$
The set of all fuzzy numbers (as given by Definition 2.1) is denoted by E^1 and is a convex cone. An alternative definition for parameter from of a fuzzy number is given by Kaleva (14).

Definition 2.2. A fuzzy number $\breve{u}$ in parametric form is a pair $(\underline{u}, \overline{u})$of function $\underline{u}(\alpha), \overline{u}(\alpha)$, $0 \le \alpha \le 1$, which satisfies the following requiremenst:

i) $\underline{u}(\alpha)$is a bounded left continuous non- decreasing function over (0, 1)
ii) $\overline{u}(\alpha)$is a bounded left continuous non- increasing function over (0, 1)
iii) $\underline{u}(\alpha) \le \overline{u}(\alpha), 0 \le \alpha \le 1$

Definition 2.3. .For arbitrary fuzzy $u = (\underline{u}(\alpha), \overline{u}(\alpha))$,$v = (\underline{v}(\alpha),\overline{v}(\alpha))$, , $0 \le \alpha \le 1$ and scalar k, we define addition, subtraction, scalar product by k and multiplication are respectively as following:

1 addition : $\underline{(u+v)}(\alpha)= (\underline{u}(\alpha)+\underline{v}(\alpha))$, $\overline{(u+v)}(\alpha) = (\overline{u}(\alpha)+\overline{v}(\alpha))$,
2 subtraction : $\underline{(u \quad v)}(\alpha)= (\underline{u}(\alpha)-\underline{v}(\alpha))$, $\overline{(u \quad v)}(\alpha) = (\overline{u}(\alpha)-\overline{v}(\alpha))$,
3 scalar product :

$$k\breve{u}=\begin{cases}\left(k\underline{u}(\alpha), k\overline{u}(\alpha)\right), & k \ge 0\\ \left(k\underline{u}(\alpha), k\overline{u}(\alpha)\right), & k < 0\end{cases} \quad \ldots\ldots(1)$$

4- multiplication:

$$\tilde{u} \quad \tilde{v} = \begin{cases}\underline{uv}(\alpha) = \max\{\underline{u}(\alpha)\underline{v}(\alpha), \underline{u}(\alpha)\overline{v}(\alpha), \overline{u}(\alpha)\underline{v}(\alpha), \overline{u}(\alpha)\overline{v}(\alpha)\}\\ \overline{uv}(\alpha) = \min\{\underline{u}(\alpha)\underline{v}(\alpha), \underline{u}(\alpha)\overline{v}(\alpha), \overline{u}(\alpha)\underline{v}(\alpha), \overline{u}(\alpha)\overline{v}(\alpha)\}\end{cases} \quad \ldots\ldots(2)$$

Defined 2.4.For arbitrary Fuzzy numbers $\tilde{u}, \tilde{v} \in E^1$

$$D(\tilde{u}, \tilde{v}) = \max\{\sup_{\alpha\in[0,1]} \left|\underline{u}(\alpha) \quad \underline{v(\alpha)}\right| , \sup_{\alpha\in[0,1]} |\overline{u}(\alpha) \quad \overline{v}(\alpha)|\}, \quad \ldots\ldots(3)$$

In the distance between The$\tilde{u}$and$\tilde{v}$, it is prove (E^1, D) is a complete metric space.

Definition 2.5. The integral of a fuzzy function was define in (14) by using the Riemann integral concept.

Let : $[a, b] \to E^1$. For Fuzzy function, for each partition p={t0, …, tn} of $[a, b]$ and for arbitrary $\xi_i \in [t_{i-1}, t_i]$, $1 \le i \le n$, suppose
$R_P=\sum_{i=1}^{n} f(\xi_i)(t_i \quad t_{i-1})$
……(4)

$\max\{|t_i \quad t_{i-1}|, 1 \le i \le n\}$.

The define integral of f(t) over (a, b) is

$\int_a^b f(t)dt = \lim_{\to 0} R_P$, ……………………………..(5)

If the fuzzy function f(t) is continuous in metric D,its definite the integral exists and also

$(\int_a^b \underline{f(t;\alpha)dt}) = \int_a^b \underline{f}(t;\alpha)dt, \quad (\int_a^b \overline{f(t;\alpha)}dt) = \int_a^b \overline{f}(t;\alpha)dt$ ……………………………(6)

It should be noted that the fuzzy integral can be also defined using the Lebesgue – type approach. However, if f(t) is continuous, both approaches yield the same value .More details about the properties of the fuzzy integral

3.Proposal fuzzy nonlinear volterra integral classes equation

The fuzzy nonlinear integral equation with integral kernel which is discussed in this work is the fuzzy nonlinear Volterra integral equation (FNVIE2) as follows:

$$\tilde{u}(x) = \tilde{f}(x) + \lambda \int_a^x k\left(x,t,\tilde{k}_1\left(t,\tilde{F}(t,\tilde{u}(t))\right)\right)dt \quad \text{……………………………(7)}$$

where $\lambda \geq 0, \tilde{f}(x)$a fuzzy function of x : a ≤ x ≤ b, $k(x, t\tilde{k}_1(\tilde{F}(t,\tilde{u}(t))))$ is analytic functions $on[a,b]and\tilde{F}(t,\tilde{u}(t))$ are nonlinear function on (a,b). For solving in parametric form of Eq. (7), consider $(\underline{f}(x,\alpha),\overline{f}(x,\alpha))$ and $(\underline{u}(x,\alpha),\overline{u}(x,\alpha))$ and, $0 \leq \alpha \leq 1$ and t ,s∈(a,b) are parametric form of $\tilde{f}(x)$and $\tilde{u}(x)$, respectively. then, parametric form of Eq. (7) is as follows

$$\underline{u}(x,\alpha) = \underline{f}(x,\alpha) + \underline{\int_a^x k\left(x,t,k_1\left(t,F(t,u(t,\alpha))\right)\right)dt} \quad \text{…………………………...(8)}$$

$$\overline{u}(x,\alpha) = \overline{f}(x,\alpha) + \overline{\int_a^x k\left(x,t,k_1\left(t,F(t,u(t,\alpha))\right)\right)dt}$$

For each 0≤α≤1 and a≤x≤b. We can see that Eq. (7) convert to a system of nonlinear Volterra integral equations in crisp case for each 0 ≤ α≤ 1 and a ≤ t ≤ b. Now, we explain analysis perturbation methods as approximating solution of this system of nonlinear integral equations in crisp case. then, we find approximate solutions for $\tilde{u}$(x), a ≤ x ≤ b
We write the system (7) we obtain

$$\underline{u}(x,\alpha) = \underline{f}(x,\alpha) + \int_a^c k(x,t,\underline{k_1\left(t,F(t,u(t,\alpha))\right)}) + \int_c^x k(x,t,\overline{k_1\left(t,F(t,u(t,\alpha))\right)}) \quad \text{…………………………(9)}$$

$$\overline{u}(x,\alpha) = \overline{f}(x,\alpha) + \int_a^x k(x,t,\overline{k_1\left(t,F(t,u(t,\alpha))\right)}) + \int_c^x k(x,t,\underline{k_1\left(t,F(t,u(t,\alpha))\right)})$$

where$0 \leq t \leq c \ , c \leq t \leq x \ , 0 \leq \alpha \leq 1$

Now we will find the parameter for the Eq(7) , as follows

$$\underline{k(x,t,k_1\left(t,F(t,u(t,\alpha))\right))} = \begin{cases} k(x,t,\underline{k_1\left(t,F(t,u(t,\alpha))\right)}), & \underline{k(x,t,k_1\left(t,F(t,u(t,\alpha))\right))} \geq 0 \\ k(x,t,\overline{k_1\left(t,F(t,u(t,\alpha))\right)}), & \underline{k(x,t,k_1\left(t,F(t,u(t,\alpha))\right))} < 0 \end{cases}$$

$$\underline{k_1\left(t,F(t,u(t,\alpha))\right)} = \begin{cases} \underline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right), \underline{k_1\left(t,F(t,u(t,\alpha))\right)} \geq 0 \\ \underline{k_1}(t,\overline{F}(t,\overline{u}(t,\alpha)), \ \underline{k_1\left(t,F(t,u(t,\alpha))\right)} < 0 \end{cases}$$

$$\overline{k_1\left(t,F(t,u(t,\alpha))\right)} = \begin{cases} \overline{k_1}(\overline{F}(t,\overline{u}(t,\alpha)), & \overline{k_1\left(t,F(t,u(t,\alpha))\right)} \geq 0 \\ \overline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right), & \overline{k_1\left(t,F(t,u(t,\alpha))\right)} < 0 \end{cases}$$

$$\overline{k\left(x,t,k_1\left(t,F(t,u(t,\alpha))\right)\right)} = \begin{matrix} k(x,t,\overline{k_1\left(t,F(t,u(t,\alpha))\right)}) \ , \overline{k\left(x,t,k_1\left(t,F(t,u(t,\alpha))\right)\right)} \geq 0 \\ k(x,t,\underline{k_1\left(t,F(t,u(t,\alpha))\right)}), \overline{k\left(x,t,k_1\left(t,F(t,u(t,\alpha))\right)\right)} < 0 \end{matrix}$$

$$\overline{k_1\left(t,F\left(t,u(t,\alpha)\right)\right)}=\begin{cases}\overline{k_1}(\overline{F}(t,\overline{u}(t,\alpha)), & \overline{k_1\left(t,F\left(t,u(t,\alpha)\right)\right)}\geq 0\\ \overline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right), & \overline{k_1\left(t,F\left(t,u(t,\alpha)\right)\right)}<0\end{cases}$$

$$\underline{k_1\left(t,F\left(t,u(t,\alpha)\right)\right)}=\begin{cases}\underline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right), & \underline{k_1\left(t,F\left(t,u(t,\alpha)\right)\right)}\geq 0\\ \underline{k_1}(t,\overline{F}(t,\overline{u}(t,\alpha)), & \underline{k_1\left(t,F\left(t,u(t,\alpha)\right)\right)}<0\end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

Than

$$\underline{u}(x,\alpha)=\underline{f}(x,\alpha)+\int_a^d k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right)dt+\int_d^c k(x,t,\overline{k_1}(\overline{F}(t,\overline{u}(t,\alpha))dt+\int_c^e k(x,t,\underline{k_1}(t,\overline{F}(t,\overline{u}(t,\alpha))dt + \int_e^x k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right)dt$$

$$\overline{u}(x,\alpha)=\overline{f}(x,\alpha)+\int_a^d k(x,t,\overline{k_1}(\overline{F}(t,\overline{u}(t,\alpha))+\int_d^c k(x,t,k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right)+\int_e^e k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right)dt +\int_e^x k(x,t,\underline{k_1}(t,\overline{F}(t,\overline{u}(t,\alpha))dt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11)$$

where $0\leq t\leq d$, $d\leq t\leq c$, $c\leq t\leq e$, $e\leq t\leq x$, $0\leq\alpha\leq 1$

3.1 Homotopy Perturbation Method

Consider the fuzzy nonlinear volterra integral equation of the second kind

$$\tilde{u}(x)=\tilde{f}(x)+\lambda\int_a^x k\left(x,t,\tilde{k}_1\left(t,\tilde{F}\left(t,\tilde{u}(t)\right)\right)\right)dt$$

Let

$$L(\underline{u})=\underline{u}(x,\alpha)\quad \underline{f}(x,\alpha)\quad \int_a^d k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right)dt\quad \int_d^c k(x,t,\overline{k_1}(\overline{F}(t,\overline{u}(t,\alpha))dt\quad \int_c^e k(x,t,\underline{k_1}(t,\overline{F}(t,\overline{u}(t,\alpha))dt\quad \int_e^x k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right)dt=0$$

$$L(\overline{u})=\overline{u}(x,\alpha)\quad \overline{f}(x,\alpha)\quad \int_a^d k(x,t,\overline{k_1}(\overline{F}(t,\overline{u}(t,\alpha))\quad \int_d^c k(x,t,k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right)\quad \int_e^e k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right)dt\quad \int_e^x k(x,t,\underline{k_1}(t,\overline{F}(t,\overline{u}(t,\alpha))dt=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12)$$

then we defined the homotopy

$H(\underline{u},p),H(\overline{u},p)$by

$$\begin{cases}H(\underline{u},0)=F(\underline{u}), & H(\underline{u},1)=L(\underline{u})\\ H(\overline{u},0)=F(\overline{u}), & H(\overline{u},1)=L(\overline{u})\end{cases}$$

where$F(\underline{u})$, $F(\overline{u})$ are functional operators with solution say $\underline{u}_0$, $\overline{u}_0$ which can be obtained easily. We choose a convex homotopy

$$H(\underline{u},p)=(1\quad p)F(\underline{u})+pL(\underline{u})=0$$
$$H(\overline{u},p)=(1\quad p)F(\overline{u})+pL(\overline{u})=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13)$$

and continuously trace an implicitly defined curve from a starting points $H(\underline{u}_0,0),H(\overline{u}_0,0)$ to a solution $H(\underline{u},1),H(\overline{u},1)$. The embedding parameter p monotonically increases from 0 to 1 as the trivial problem $F(\underline{u})=0$, $F(\overline{u})=0$ continuously deformed to the original problem $L(\underline{u})=0$,

$L(\overline{u})=0$. The parameter can be considered as an expanding parameter. In fact HPM uses the homotopy parameter p as an expanding parameter to obtain

$$\underline{u}=\underline{u}_0+p\underline{u}_1+p^2\underline{u}_2+\quad\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14)$$
$$\overline{u}=\overline{u}_0+p\overline{u}_1+p^2\overline{u}_2+\quad\ldots.$$

when $p \to 1$, (7) corresponding to (6) and gives an approximate as follows

$$\underline{u} = \lim_{p\to 1} \underline{u}_2 + \quad \ldots$$

$$\overline{u} = \lim_{p\to 1} \overline{u}_2 + \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(15)$$

The series (8) converges in most cases, and the rate of convergence depends on $L(\underline{u})$, $L(\overline{u})$

Now we applied the HPM for solving system for fuzz volterra nonlinear integral equation

$$F(\underline{u}) = \underline{u}(x,\alpha) \quad \underline{f}(x,\alpha)$$
$$F(\overline{u}) = \overline{u}(x,\alpha) \quad \overline{f}(x,\alpha)$$

$$H(\underline{u},p) =$$
$$\underline{u}(x,\alpha) \quad \underline{f}(x,\alpha) \quad \int_a^d k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right)dt \quad \int_d^c k(x,t,\overline{k_1}(\overline{F}\left(t,\overline{u}(t,\alpha)\right)dt \quad \int_c^e k(x,t,\underline{k_1}(t,\overline{F}\left(t,\overline{u}(t,\alpha)\right)dt$$
$$\int_e^x k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right)dt = 0$$

$$H(\overline{u},p) =$$

$$\overline{u}(x,\alpha) \quad \overline{f}(x,\alpha) \quad \int_a^d k(x,t,\overline{k_1}(\overline{F}\left(t,\overline{u}(t,\alpha)\right) \quad \int_d^c k(x,t,k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right) \quad \int_e^e k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right)dt$$
$$\int_e^x k(x,t,\underline{k_1}(t,\overline{F}\left(t,\overline{u}(t,\alpha)\right)dt = 0$$

and equation the term with identical power of p , we obtain

$$\begin{cases} p^0: \underline{u}_0(x,\alpha) \quad \underline{f}(x,\alpha) = 0 \qquad \underline{u}_0(x,\alpha) = \underline{f}(x,\alpha) \\ p^0: \overline{u}_0(x,\alpha) \quad \overline{f}(x,\alpha) = 0 \qquad \overline{u}_0(x,\alpha) = \overline{f}(x,\alpha) \end{cases}$$

$$p^1: \underline{u}_1(x,\alpha) \quad \int_a^d k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u_0}(t,\alpha)\right)dt \quad \int_d^c k(x,t,\overline{k_1}(\overline{F}\left(t,\overline{u_0}(t,\alpha)\right)dt \quad \int_c^e k(x,t,\underline{k_1}(t,\overline{F}\left(t,\overline{u_0}(t,\alpha)\right)dt$$
$$\int_e^x k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u_0}(t,\alpha)\right)dt = 0$$

$$\underline{u}_1(x,\alpha) = \int_a^d k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u_0}(t,\alpha)\right)dt + \int_d^c k(x,t,\overline{k_1}(\overline{F}\left(t,\overline{u_0}(t,\alpha)\right)dt + \int_c^e k(x,t,\underline{k_1}(t,\overline{F}\left(t,\overline{u_0}(t,\alpha)\right)dt$$
$$+ \int_e^x k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u_0}(t,\alpha)\right)dt$$

$$p^1: \overline{u}_1(x,\alpha) \quad \int_a^d k(x,t,\overline{k_1}(\overline{F}\left(t,\overline{u_0}(t,\alpha)\right) \quad \int_d^c k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u_0}(t,\alpha)\right) \quad \int_e^e k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u_0}(t,\alpha)\right)dt$$
$$\int_e^x k(x,t,\underline{k_1}(t,\overline{F}\left(t,\overline{u_0}(t,\alpha)\right)dt = 0 \qquad \overline{u}_1(x,\alpha)$$
$$= \int_a^d k(x,t,\overline{k_1}(\overline{F}\left(t,\overline{u_0}(t,\alpha)\right) + \int_d^c k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u_0}(t,\alpha)\right) + \int_e^e k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u_0}(t,\alpha)\right)dt$$
$$+ \int_e^x k(x,t,\underline{k_1}(t,\overline{F}\left(t,\overline{u_0}(t,\alpha)\right)dt$$

Now we will write the general formula for HPM to solve our system

$$\underline{u}_0(x,\alpha) = \underline{f}(x,\alpha)$$

$\underline{u}_{n+1}(x,\alpha) = \int_a^d k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u_n}(t,\alpha)\right)dt + \int_d^c k(x,t,\overline{k_1}(\overline{F}(t,\overline{u_n}(t,\alpha))dt + \int_c^e k(x,t,\underline{k_1}(t,\overline{F}(t,\overline{u_n}(t,\alpha))dt +$
$\int_e^x k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u_n}(t,\alpha)\right)dt$

n=1,2,….

$\overline{u}_0(x,\alpha) = \overline{f}(x,\alpha)$

$$\overline{u}_{n+1}(x,\alpha) = \int_a^d k(x,t,\overline{k_1}(\overline{F}(t,\overline{u_n}(t,\alpha)) + \int_d^c k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u_n}(t,\alpha)\right) + \int_e^e k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u_n}(t,\alpha)\right)dt + \int_e^x k(x,t,\underline{k_1}(t,\overline{F}(t,\overline{u_n}(t,\alpha))dtn = 1,2,....$$

Then if $k(x,t)$ is non negative and non positive we get

$\underline{u}_0(x,\alpha) = \underline{f}(x,\alpha)$

$$\underline{u}_{n+1}(x,\alpha) = \int_a^d k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u_n}(t,\alpha)\right)dt + \int_d^c k(x,t,\overline{k_1}(\overline{F}(t,\overline{u_n}(t,\alpha))dt + \int_c^e k(x,t,\underline{k_1}(t,\overline{F}(t,\overline{u_n}(t,\alpha))dt + \int_e^x k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u_n}(t,\alpha)\right)dt$$

$\overline{u}_0(x,\alpha) = \overline{f}(x,\alpha)$

$$\overline{u}_{n+1}(x,\alpha) = \int_a^d k(x,t,\overline{k_1}(\overline{F}(t,\overline{u_n}(t,\alpha)) + \int_d^c k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u_n}(t,\alpha)\right) + \int_e^e k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u_n}(t,\alpha)\right)dt + \int_e^x k(x,t,\underline{k_1}(t,\overline{F}(t,\overline{u_n}(t,\alpha))dtn = 1,2,....$$

3.1 Fuzzy integration of a crisp (real- valued) function over a fuzzy interval(Dubois, 1982a)

We shall consider a case for which Dubois and Prade(Dubois, 1982a) have proposed a fuzzy domain D of the real line R assumed to be delimited by two bounds $\tilde{A} and \tilde{B}$ in the following sense

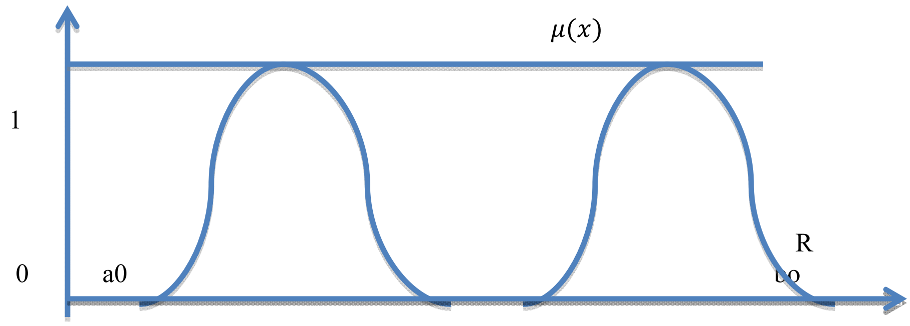


Fig. 1. Crisp valued function over a fuzzy interval

$\tilde{A} and \tilde{B}$are fuzzy sets on R, whose membership function are $\mu_{\tilde{A}}$ and $\mu_{\tilde{B}}$ repectively , from R to (0,1).

$\forall x \in R$,$\mu_{\tilde{A}}(x)(respectively\mu_{\tilde{B}}(x))$ evaluates to what extent x can be considered as a greatest lower bound (respectively least upper bound) of D

$\tilde{A} and \tilde{B}$are normalized fuzzy sets.

$\tilde{A} and \tilde{B}$ are convex fuzzy set

$Disdenoed\ (\tilde{A},\tilde{B})$, $\tilde{A} and \tilde{B}$ are assumed ordered in the sense that

$a_0 = InfS(\tilde{A}) \leq SupS(\tilde{B}) = b_0$

$w\ \ ereS(.)$stands for support

Definition 2 :Let$f(u)$ be a real- valued mapping, and integerable on the interval

$1 = [InfS(\tilde{A}), SupS(\tilde{B})], t\ \ ent\ \ eintegralof$ f(u) over the domain de-limited is defined according to the extension principle by:

$\mu_{I(\tilde{x},\tilde{y})}(Z) = Sup_{x,y\in I: z=\int_x^y f(u)du} \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)\}$ ………………………………………..(12)

To develop an applicable numerical algorithm for computing fuzzy integration , it is very important to discuss the following useful remarks and propositions.

Remarks 1:

1-If one of he bounds is not fuzzy , we consider the integral of $f(u)$ over $[a, \tilde{B})asI(a, \tilde{B})$ and it's membership function can be defined as (Dubois , 1980)

$\mu_{I(a,\tilde{B})}(z) = Sup_{y:z=\int_a^y f(u)du} \mu_B(y)$ …………………………………………(13)
$= Sup_{y:F(y)-F(a)} \mu_B(y)$

where F is an anti-derivative of f

2- If both bounds are fuzzy, then (2.1) can be rewritten as:

$\mu_{I(\tilde{A},\tilde{B})}(z) = Sup_{z=F(y)-F(x)} \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)\}$

$= Sup_{x\in R} \min\{\mu_{\tilde{A}}(x), Sup_{y:z=F(y)-F(x)} \mu_{\tilde{B}}(y)\}$ ……………………………………..(14)

$= Sup_{x\in R} \min\{\mu_{\tilde{A}}(x), \mu_{I(a,\tilde{B})}(Z)\}$

The following are some of the useful properties

Proposition 1.(Dubois, 1982a)

Let $F(x)$ be an anti-derivative function of f(x), i.e, $F(x) = \int_c^x f(x)dx$, for some $c \in I$(the interval of integration) denoted $F(\tilde{A})$ is the image of the fuzzy set $\tilde{A}t\ \ roug\ \ F$, defined by the extension principle $\forall z \in R$,

$\mu_{F(\widetilde{A})}(Z) = Sup_{x:z=F(x)} \mu_{\tilde{A}(x)}$.

Moreover, denotes extended principle the following proposition holds:

$\int_{\tilde{A}}^{\tilde{B}} f = F(\tilde{B}) \quad f(\tilde{A})$ …………………………………….(15)

Proposition 2. (Dubois, 1982a)

Let $fandg$ be two real mapping integerable on interval I, $(f: I \to R, g: I \to R)$ then :

$\int_{\tilde{A}}^{\tilde{B}}(f + g) \subseteq \int_{\tilde{A}}^{\tilde{B}} f \oplus \int_{\tilde{A}}^{\tilde{B}} g$ …………………………………(16)

where⊆ denotes the usual fuzzy set inclusion , ⊕ denotes the extended additionproposition

3.(Dubois, 1982a)

If $fandg$ are both either positive or negative integerable real mapping

$(f: I \to R^+, g: I \to R^+)or(f: I \to R^-, g: I \to R^-)$, then the equality holds , i.e

$\int_{\tilde{A}}^{\tilde{B}}(f + g) = \int_{\tilde{A}}^{\tilde{B}} f \oplus \int_{\tilde{A}}^{\tilde{B}} g$ …………………………………………..(17)

Proposition 4.(Dubois, 1982a)

Let $\acute{D} and \acute{D}$ be domains of R delimited by fuzzy bounds $(\tilde{A},\tilde{C})$ and $(\tilde{C},\tilde{B})$respectively, then for any integerable mapping

$\int_D f \subseteq \int_D f \quad \int_D f$ ………………………………………….(18)

where D is delimited by $(\tilde{A},\tilde{B})$, the equality holds if and only if $\tilde{C}$ is real number

Proof:

$$\int_D f = F(\tilde{B}) \quad F(\tilde{A})$$

$$\int_D f \quad \int_D f = (F(\tilde{B}) \quad F(\tilde{C})) \oplus (F(\tilde{C}) \quad F(\tilde{A}))$$

Not that $F(\tilde{C}))$ $(F(\tilde{C}) = 0$ if and only if $\tilde{C}$ is real number, otherwise

$V(z) = Sup_{z=u-v+w-t} \min\{\mu_{F(\widetilde{B})}(u), \mu_{F(\widetilde{C})}(v), \mu_{F(\tilde{c})}(w), \mu_{F(\tilde{A})}(t)\}\}$

$\geq Sup_{z=u-t} \min\left(\mu_{F(\widetilde{B})}(u), \mu_{F(\tilde{A})}(t)\right) = \mu(z)$

Since we add the constraint $v = w$, and we can drop $F(\tilde{C})$ which is normalized

Remarks 5.(Dubois, 1980)

$I(a,\tilde{B}) = F(\tilde{B}) \quad F(a)$ is the value of the extended $F(x) \quad F(a)$ $,w \ \ enx = \tilde{B}$
Proof :

Let μ be the membership function of $I(a,\tilde{B}) andvbet \ \ emembers \ \ ipfunction$

$F(\tilde{B}) \quad F(a)$

$V(z) = Sup_{u,v:z=u-v} \min\{Sup_{a:u=F(a)}\mu_a(a), Sup_{y:v=F(y)}\mu_{\tilde{B}}(y)\}$
$= Sup_{u,v:z=u-v} \min\{\mu_a(a), Sup_{y:v=F(y)}\mu_{\tilde{B}}(y)\}$
$= Sup_{a,y:F(y)-F(a)=z} \min\{\mu_a(a), \mu_{\tilde{B}}(y)\}$

where

$$F(y) \quad F(a) = \int_a^y f(x)dx$$

$I(\tilde{A},\tilde{B})$ is the fuzzy value of the extended fuzzifying function :

$y = F(B) \quad F(x), \qquad for X = \tilde{A}$

3.2. Numerical fuzzy integration of a crisp function over a fuzzy interval

We shall develop a computational algorithm for computing a numerical fuzzy integration of crisp function over a fuzzy interval. The basic principle of this technique for continuous fuzzy number. The continuous fuzzy number is discretized and then converted into discrete fuzzy number so that the numerical fuzzy integration over the discrete case can be easily implemented .
Mathematically, we can represent the fuzzy integration:

$$I(\tilde{A},\tilde{B}) = \{\int_{\tilde{A}}^{\tilde{B}} f(u)du | \tilde{A} and \tilde{B} are fuzzy numbers and f(u) is integrable$$

valued real function from R to R}

For discrete fuzzy numbers, we develop the following

3.2.1 Fuzzy Integration for Discrete Fuzzy Numbers

Ageneral procedure for computing the fuzzy integration for discrete fuzzy numbers

$\tilde{A}$ and $\tilde{B}$ have been developed and as follows:

Let the universe set be $X, w\ \ en\tilde{A}and\tilde{B}$ are discrete fuzzy numbers,

Then

$$\tilde{A} = \sum_{i=1}^{n} \mu_{\tilde{A}}(x_i)/x_i \quad , \qquad x_i \in X$$

and

$$\tilde{B} = \sum_{i=1}^{m} \mu_{\tilde{B}}(y_i)/y_i \quad , \qquad y_i \in X$$

Since $\tilde{A}and\tilde{B}$ are non empty fuzzy sets, then there are $n_1 and n_2$ positive integer numbers, such that the support of the fuzzy sets $\tilde{A}and\tilde{B}$ can be given by :

$$S(\tilde{A}) = \{x_1, x_2, \dots \dots, x_{n1}\} and S(\tilde{B}) = \{x_1, x_2, \dots \dots, x_{n2}\}$$

Let

$$S_1 = S(\tilde{A}) and S_2 = S(\tilde{B})$$

Then define :

$$S_1 \times S_2 = \{(x, y_j) | x_i \in S_1, y_j \in S_2\} \ , \qquad i = 1,2, \dots\dots n_1, j = 1,2, \dots\dots, n_2$$

Then let

$$I_{s_1 \times s_2} = \int_{s_1 \times s_2} f(u)du = \int_{x_i}^{y_j} f(u)du \qquad \dots\dots (19)$$

It should be noted that (19) is fuzzy set according to (2.1) where its membership function can be defined as:

$$\mu_{Is_1 \times s_2} = Sup_{x,y \in s_1 \times s_2} \min\{\mu_{\tilde{A}}(x_i), \mu_{\tilde{B}}(y_j)\}, i = 1,2, \dots\dots, n_1 \ , j = 1,2, \dots\dots, n_2$$

Let us define

$$I_{i,j} = \int_{x_i}^{y_j} f(u)du$$

If $I_{i_1,j_1} = I_{i_2,j_2}, i_1 \neq i_2, j_1 \neq j_2$. Then the supermum of this membership function of

I_{i_1,j_1}, I_{i_2,j_2}have been taken. Otherwise, no need for the supremum operation. Thus the fuzzy integration is:

$$I(\tilde{A}, \tilde{B}) = \{(I_{s_1 \times s_2}, \mu(I_{s_1 \times s_2})) | I_{s_1 \times s_2} = \int_{s_1 \times s_2} f(u)du\}$$

Remark 2:

When $I_{i,j} = \int_{x_i}^{y_j} f(u)du = \int_{y_i}^{x_j} f(u)du$, fuzzy integrals are fuzzy sets with membership function

$$\mu_{\int_{x_i}^{y_j} f(u)du}(u) = \mu_{\int_{y_i}^{x_j} f(u)du}(\ u)$$

3.2.2. Fuzzy integration for continuous fuzzy numbers

For continuous membership function of the fuzzy numbers $\tilde{A}and\tilde{B}$ one can develop the following.
When $\tilde{A}and\tilde{B}$are continuous fuzzy number then

$\tilde{A} = \int_x \mu_{\tilde{A}}(x)\backslash x , \quad x \in X$

$\tilde{B} = \int_x \mu_{\tilde{B}}(y)\backslash y , \quad y \in X$

Let us define the L.R type to represent the fuzzy numbers $\tilde{A} and \tilde{B}$, where the membership function are defined as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} L\left(\frac{m \quad x}{\alpha}\right), & x \leq m \\ R\left(\frac{x \quad m}{\beta}\right), & x \geq m \end{cases}$$

…………………….(20)

$$\mu_{\tilde{B}}(y) = \begin{matrix} L\left(\frac{\acute{m} \quad y}{\acute{\alpha}}\right), & y \leq \acute{m} \\ R\left(\frac{x \quad \acute{m}}{\acute{\beta}}\right), & y \geq \acute{m} \end{matrix}$$

Then

$\tilde{A} = (m, \alpha, \beta)_{LR}$ and
$\tilde{B} = (\acute{m}, \acute{\alpha}, \acute{\beta})_{LR}$

Discretization of the above continuous fuzzy numbers $\tilde{A} and \tilde{B}$ can be done in two ways, and as follows:

3.2.2.1. Discretization of a abscissa ($x \quad axis$)

Let $\tilde{A}(\tilde{B})$ be a continuous fuzzy number and $m(\acute{m})$ be the mean value of the fuzzy number $\tilde{A}(\tilde{B})$ with a membership value $\mu_{\tilde{A}}(m) = 1, (\mu_{\tilde{B}}(\acute{m}) = 1), \alpha and \beta (\acute{\alpha} and \acute{\beta})$ are left and right spreads of $\mu_{\tilde{A}}(x)(\mu_{\tilde{B}}(x))$, the membership function of the continuous fuzzy number have unsharp boundaries. Furthermore , the reference function L (or R) for the fuzzy number is decreasing on $)0, +\infty(, t \quad usat x \to \infty , t \quad en \mu_{\tilde{A}}(x) \to 0$

Step 1

Since the domain of the membership function which depending on LR-type , in general, it's not bounded, so to find the lower limit and the upper limit for the reference function L and R respectively to find the boundaries of the membership function, then we have
If $x \leq m(y \leq \acute{m}), t \quad emembers \quad ipfunctionoft \quad efuzzynumber \tilde{A}(\tilde{B}) isdefineas$

$$\mu_{\tilde{A}}(x) = L\left(\frac{m \quad x}{\alpha}\right) and \mu_B(t) = L\left(\frac{\acute{m} \quad y}{\acute{\alpha}}\right)$$

Since the membership function$\mu_{\tilde{A}}(x)(\mu_B(y))$ is continuous differentiable and decreasing function, therefore, there exists $p_1 \in R(q_1 \in R)$, such that

$|\mu_{\tilde{A}}(p_1)| < \xi_1and|\mu_{\tilde{B}}(q_1)| < \xi_1$

$Forsomesmallpositiverealnumber \xi_1 and \xi_1 \ , T \quad enp_1(q_1)$ is called the lower bounds.

Similarly, if $x \geq m(y \geq \acute{m}), t \quad emembers \quad ipfunctionoft \quad efuzzynumber \tilde{A}(\tilde{B}) isdefineas$

$$\mu_{\tilde{A}}(x) = R\left(\frac{x \quad m}{\beta}\right) and \mu_B(y) = R\left(\frac{y \ ' \ m}{\acute{\beta}}\right)$$

Since the membership function$\mu_{\tilde{A}}(x)(\mu_B(y))$ is continuous differentiable and decreasing function, therefore, there exists $p_2 \in R(q_2 \in R)$, such that

$|\mu_{\tilde{A}}(p_2)| < \xi_2and|\mu_{\tilde{B}}(q_2)| < \xi_2$

$Forsomesmallpositiverealnumber\xi_2 and \xi_2$ $,T \quad enp_1(q_1)$ is called the upper bounds

Step 2:

A partition for the continuous fuzzy number $\tilde{A}$ centered at the mean v , m can be implemented and as follows:

$\Delta p = \frac{|m-p_1|}{N_1}$ (step length for left interval, where $x \leq m$)
$\Delta p \;\; = \frac{|p_2-m|}{N_2}$ (step length for left interval, where $x \geq m$)

Where

p_1= lower limit of the membership function of $\tilde{A}$ of step 1
p_2= upper limit of the membership function of $\tilde{A}$ of step 1
m= mean value of the fuzzy number $\tilde{A}$
N_1, N_2= l arge positive number

Similarly a partition for the continuous fuzzy number $\tilde{B}$ centered at the mean $\acute{m}$ divided into two partitions:

$\Delta q = \frac{|\acute{m}-q_1|}{M_1}$ (step length for left interval, where $x \leq \acute{m}$)
$\Delta p \;\; = \frac{|q_2-\acute{m}|}{M_2}$ (step length for left interval, where $x \geq \acute{m}$)

Where

q_1= lower limit of the membership function of $\tilde{B}$.
q_2= upper limit of the membership function of $\tilde{B}$.
$\acute{m}$= mean value of the fuzzy number $\tilde{B}$
M_1, M_2= large positive number

Step 3:

Let $x_0 = m\big(meanvalueoft \quad efuzzynumber\tilde{A}\big)$, then. For the left side discretized of membership function of the fuzzy number $\tilde{A}$, we have :

$x_i = x_0 \quad i \;\; p, \quad i = 0,1,2, \ldots, N_1$

with a membership function

$\mu_{\tilde{A}}(x_i) = L(\frac{x_0 \quad x_i}{\alpha})$

For the right side discretized of membership function of the fuzzy number $\tilde{A}$, we have :

$x_j = x_0 + j \;\; p \; , \quad j = 1,2, \ldots, N_2$

with a membership function

$\mu_{\tilde{A}}(x_i) = R(\frac{x_j \quad x_0}{\beta})$

Thus the approximate discrete fuzzy number for continuous fuzzy number $\tilde{A}$ can be rewritten as :

$\tilde{A} \;\; = \{(x_{N_1}, \mu(x_{N_1})), \ldots, (x_i, \mu(x_i)), \ldots, (x_0, 1), (x_1, \mu(x_1)), \ldots (x_j, \mu(x_j)), \ldots, (x_{N_2}, \mu(x_{N_2}))\}$

Similarly, let $y_0 = \acute{m}$ (mean value of the membership function of $\tilde{B}$), then. For the left side of membership function of the fuzzy number $\tilde{B}$, we have :

$y_i = y_0 \quad i \;\; q, \quad i = 0,1,2, \ldots, M_1$

with a membership function

$$\mu_{\tilde{B}}(y_i) = L(\frac{y_0 \quad y_i}{\acute{\alpha}})$$

For the right side discretized of membership function of the fuzzy number $\tilde{B}$, we have :

$$y_j = y_0 + j \quad q \;, \quad j = 1,2, \dots., M_2$$

with a membership function

$$\mu_{\tilde{B}}(y_i) = R(\frac{y_j - y_0}{\acute{\beta}})$$

the approximate discrete fuzzy number for continuous fuzzy number $\tilde{B}$ can be rewritten as :

$$\tilde{B} \;= \{(y_M, \mu(y_{M_1})), \dots.., (y_j, \mu(y_j)), \dots., (y_0, 1), (y_1, \mu(y_1)), \dots. (y_i, \mu(y_i)), \dots., (x_y, \mu(y_{M_2}))\}$$

Step 4:

Take the support for the fuzzy number $\tilde{A}$ $and \tilde{B}$, then we have

$$S(\tilde{A}\;) = \{x_0, x_1, \dots., x_{N_1+N_2}\}$$

$$S(\tilde{B}\;) = \{y_0, y_1, \dots., y_{M_!+M_2}\}$$
$Let S_1 = S(\tilde{A}\;) \;, \qquad S_2 = S(\tilde{B}\;). Define :$
$S_1 \times S_2 = \{(x_i, y_j) | x_i \in S_1 \;, y_j \in S_2, i = 0,1, \dots. N_1 + N_2, j = 0,1,2, \dots, M_1 + M_2\}$

Step 5:

For each $(x, y) \in S_1 \times S_2$,calculate the integration

$$I_{S_1 \times S_2} = \int_{S_1 \times S_2} f(u) du$$

as has been discussed

$$\mu_{IS_1 \times S_2} = Sup_{(x,y) \in S_1 \times S_2} \min \{\mu_{\tilde{A}} \;, \mu_{\tilde{B}}\;\}$$

Step 6:

Check if $I_{i_1,j_1} = I_{i_2,j_2}, i_1 \neq i_2, j_1 \neq j_2$, then the supremum of this membership function of I_{i_1,j_1}, I_{i_2,j_2} have been take

Step 7:

Then the total fuzzy integration is :

$I(\tilde{A}, \tilde{B}) = \{(I_{i,j}, \mu(I_{i,j}) | i = 0,1, \dots., K_1, j = 0,1, \dots., K_2\}$for some positive integers $K_1 and K_2$.

4. Solution of proposal fuzzy nonlinear integral classes equations

Our treatment of Crisp nonlinearvolterra integral equation classes mainly on illustrations of the known methods of finding exact, or numerical solution. In this work we present new techniques for solving crisp nonlinear volterra integral equations over the fuzzy interval. Re-writ equation (1)

$$\tilde{u}(x) = \tilde{f}(x) + \lambda \int_a^x k\left(x, t, \tilde{k}_1\left(t, \tilde{F}(t, \tilde{u}(t))\right)\right) dt$$

where $\lambda \geq 0, \tilde{f}(x)$a fuzzy function of x : a ≤ x ≤ b, $k(x, t\tilde{k}_1(\tilde{F}(t, \tilde{u}(t))))$ is analytic functions $on[a, b] and \tilde{F}(t, \tilde{u}(t))$ are nonlinear function on (a,b). For solving in parametric form of Eq. (1), consider $(\underline{f}(x, \alpha), \overline{f}(x, \alpha))$ and $(\underline{u}(x, \alpha), \overline{u}(x, \alpha))$ and, $0 \leq \alpha \leq 1$ and t ,s ∈ (a,b) are parametric form of $\tilde{f}(x)$and $\tilde{u}(x)$, respectively. Then, parametric form of Eq. (1) is as follows

$$\underline{u}(x,\alpha)=\underline{f}(x,\alpha)+\int_{L(a)}^{L(c)}k(x,t,\underline{k_1\left(t,F\left(t,u(t,\alpha)\right)\right)}+\int_{L(c)}^{L(x)}k(x,t,\overline{k_1\left(t,F\left(t,u(t,\alpha)\right)\right)}$$

$$\overline{u}(x,\alpha)=\overline{f}(x,\alpha)+\int_{R(a)}^{R(c)}k(x,t,\overline{k_1\left(t,F\left(t,u(t,\alpha)\right)\right)}+\int_{R(c)}^{R(x)}k(x,t,\underline{k_1\left(t,F\left(t,u(t,\alpha)\right)\right)}$$

where$L(0)\le t\le L(c)$,$L(c)\le t\le L(x)$, where $R(0)\le t\le R(c)$,$R(c)\le t\le R(x) 0\le\alpha\le 1$

than

$$\underline{u}(x,\alpha)=\underline{f}(x,\alpha)+\int_{L(a)}^{L(d)}k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right)dt+\int_{L(d)}^{L(c)}k(x,t,\overline{k_1}(\overline{F}\left(t,\overline{u}(t,\alpha)\right)dt+\int_{L(c)}^{L(e)}k(x,t,\underline{k_1}(t,\overline{F}\left(t,\overline{u}(t,\alpha)\right)dt$$
$$+\int_{L(e)}^{L(x)}k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right)dt$$

$$\overline{u}(x,\alpha)=\overline{f}(x,\alpha)+\int_{R(a)}^{R(d)}k(x,t,\overline{k_1}(\overline{F}\left(t,\overline{u}(t,\alpha)\right)+\int_{R(d)}^{R(c)}k(x,t,k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right)+\int_{R(c)}^{R(e)}k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right)dt$$
$$+\int_{R(e)}^{R(x)}k(x,t,\underline{k_1}(t,\overline{F}\left(t,\overline{u}(t,\alpha)\right)dt$$

……………………………(21)

where$L(0)\le t\le L(d)$, $L(d)\le t\le L(c)$, $L(c)\le t\le L(e)$, $L(e)\le t\le L(x)$,

$R(0)\le t\le R(d)$, $R(d)\le t\le R(c)$, $R(c)\le t\le R(e)$, $R(e)\le t\le R(x)$, $0\le\alpha\le 1$

Remark1:

Let the$\tilde{a}=\tilde{0} and \tilde{t}=\tilde{1}$, $\tilde{a}=\tilde{0}=[L(0),R(0)] and \tilde{x}=(L(x),R(x))$

$\tilde{c}=[L(c),R(c)],\tilde{d}=[L(d),R(d)],\tilde{e}=[L(e),R(e)]$ (12)

Re-write the eq(12) with the fuzzy number formula

$\tilde{a}=\tilde{0}=[L(0),R(0)]=\left[0\quad\sqrt{1\quad\alpha},0+\sqrt{1\quad\alpha}\right]=(\quad\sqrt{1\quad\alpha},\sqrt{1\quad\alpha})$

$\tilde{x}=[L(x),R(x)]=\left[x\quad\sqrt{1\quad\alpha},x+\sqrt{1\quad\alpha}\right]=\left[x\quad\sqrt{1\quad\alpha},x+\sqrt{1\quad\alpha}\right]$

$\tilde{c}=[L(c),R(c)]=\left[c\quad\sqrt{1\quad\alpha},c+\sqrt{1\quad\alpha}\right]=\left[0.5\quad\sqrt{1\quad\alpha},0.5+\sqrt{1\quad\alpha}\right]$

$\tilde{d}=[L(d),R(d)]=\left[d\quad\sqrt{1\quad\alpha},d+\sqrt{1\quad\alpha}\right]=\left[0.25\quad\sqrt{1\quad\alpha},0.25+\sqrt{1\quad\alpha}\right]$

$\tilde{e}=[L(e),R(e)]=\left[e\quad\sqrt{1\quad\alpha},e+\sqrt{1\quad\alpha}\right]=\left[0.75\quad\sqrt{1\quad\alpha},0.75+\sqrt{1\quad\alpha}\right]$…(21)

C=0.5 , d=0.25 , e=0.75

Substituting eq(21) in eq (1), we obtain

$$\underline{u}(x,\alpha)=\underline{f}(x,\alpha)+\int_{L(a)}^{L(d)}k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right)dt+\int_{L(d)}^{L(c)}k(x,t,\overline{k_1}(\overline{F}\left(t,\overline{u}(t,\alpha)\right)dt+\int_{L(c)}^{L(e)}k(x,t,\underline{k_1}(t,\overline{F}\left(t,\overline{u}(t,\alpha)\right)dt$$
$$+\int_{L(e)}^{L(x)}k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right)dt$$

$$\overline{u}(x,\alpha)=\overline{f}(x,\alpha)+\int_{R(a)}^{R(d)}k(x,t,\overline{k_1}(\overline{F}\left(t,\overline{u}(t,\alpha)\right)+\int_{R(d)}^{R(c)}k(x,t,k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right)+\int_{R(c)}^{R(e)}k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right)dt$$

$$+\int_{R(e)}^{R(x)}k(x,t,\underline{k_1}(t,\overline{F}\left(t,\overline{u}(t,\alpha)\right)dt$$

We will use the all step in part 4 to find the solution of the system (2), . the numerical solution of the fuzzy function over the fuzzy interval using the LR-type representation of fuzzy interval. Some numerical examples are prepared to show the efficiency and accuracy of the methods

5. Numerical example

We will apply theHomotopy perturbation method to solve our problem and compared with the exact solution

Example 1

Consider the fuzzy nonlinear volterra integral equation

$$\tilde{u}(x) = \tilde{f}(x) + \lambda \int_{\tilde{a}}^{\tilde{x}} k\left(x, t, \tilde{k}_1\left(t, \tilde{F}(t, \tilde{u}(t))\right)\right) dt$$

Step 1

$$\underline{u}(x,\alpha) = \underline{f}(x,\alpha) + \int_{L(a)}^{L(c)} k(x,t,\underline{k_1\left(t, F(t,u(t,\alpha))\right)} + \int_{L(c)}^{L(x)} k(x,t,\overline{k_1\left(t, F(t,u(t,\alpha))\right)}$$

$$\overline{u}(x,\alpha) = \overline{f}(x,\alpha) + \int_{R(a)}^{R(c)} k(x,t,\overline{k_1\left(t, F(t,u(t,\alpha))\right)} + \int_{R(c)}^{R(x)} k(x,t,\underline{k_1\left(t, F(t,u(t,\alpha))\right)}$$

where$L(0) \leq t \leq L(c)$,$L(c) \leq t \leq L(x)$, where $R(0) \leq t \leq R(c)$,$R(c) \leq t \leq R(x) 0 \leq \alpha \leq 1$

than

$$\underline{u}(x,\alpha) = \underline{f}(x,\alpha) + \int_{L(a)}^{L(d)} k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right) dt + \int_{L(d)}^{L(c)} k(x,t,\overline{k_1}(\overline{F}(t,\overline{u}(t,\alpha))dt + \int_{L(c)}^{L(e)} k(x,t,\underline{k_1}(t,\overline{F}(t,\overline{u}(t,\alpha))dt + \int_{L(e)}^{L(x)} k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right) dt$$

$$\overline{u}(x,\alpha) = \overline{f}(x,\alpha) + \int_{R(a)}^{R(d)} k(x,t,\overline{k_1}(\overline{F}(t,\overline{u}(t,\alpha)) + \int_{R(d)}^{R(c)} k(x,t,k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right) + \int_{R(c)}^{R(e)} k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right) dt$$

$$+\int_{R(e)}^{R(x)} k(x,t,\underline{k_1}(t,\overline{F}(t,\overline{u}(t,\alpha))dt$$

where$L(0) \leq t \leq L(d)$, $L(d) \leq t \leq L(c)$, $L(c) \leq t \leq L(e)$, $L(e) \leq t \leq L(x)$, $R(0) \leq t \leq R(d)$, $R(d) \leq t \leq R(c)$, $R(c) \leq t \leq R(e)$, $R(e) \leq t \leq R(x), 0 \leq \alpha \leq 1$

The exact solution

$\underline{u}(x,\alpha) = x\alpha$, $\overline{u}(x,\alpha) = x(2 \quad \alpha)$

The Kernal of our problem

$k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right)$ =xt^3($\underline{u}(t,\alpha)$)^2

$k(x,t,\overline{k_1}(\overline{F}(t,\overline{u}(t,\alpha)) = xt$^2($\overline{u}(t,\alpha)$)^2

$k(x,t,\underline{k_1}(t,\overline{F}(t,\overline{u}(t,\alpha)) = xt$^3($\overline{u}(t,\alpha)$)^2

$k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right) = xt$^2($\underline{u}(t,\alpha)$)^2

Remark1:

Let the$\tilde{a} = \tilde{0}$ *and* $\tilde{t} = \tilde{1}$, $\tilde{a} = \tilde{0} = [L(0), R(0)]$ *and* $\tilde{x} = [L(x), R(x)]$

$\tilde{c} = [L(c), R(c)], \tilde{d} = [L(d), R(d)], \tilde{e} = [L(e), R(e)]$ (12)

Re-write the eq(12) with the fuzzy number formula

$\tilde{a} = \tilde{0} = [L(0), R(0)] = [0 \quad \sqrt{1 \quad \alpha}, 0 + \sqrt{1 \quad \alpha}] = [\quad \sqrt{1 \quad \alpha}, \sqrt{1 \quad \alpha}]$

$\tilde{x} = [L(x), R(x)] = [x \quad \sqrt{1 \quad \alpha}, x + \sqrt{1 \quad \alpha}] = [x \quad \sqrt{1 \quad \alpha}, x + \sqrt{1 \quad \alpha}]$

$\tilde{c} = [L(c), R(c)] = [c \quad \sqrt{1 \quad \alpha}, c + \sqrt{1 \quad \alpha}] = [0.5 \quad \sqrt{1 \quad \alpha}, 0.5 + \sqrt{1 \quad \alpha}]$

$\tilde{d} = [L(d), R(d)] = [d \quad \sqrt{1 \quad \alpha}, d + \sqrt{1 \quad \alpha}] = [0.25 \quad \sqrt{1 \quad \alpha}, 0.25 + \sqrt{1 \quad \alpha}]$

$\tilde{e} = [L(e), R(e)] = [e \quad \sqrt{1 \quad \alpha}, e + \sqrt{1 \quad \alpha}] = [0.75 \quad \sqrt{1 \quad \alpha}, 0.75 + \sqrt{1 \quad \alpha}]$ ....(21)

C=0.5 , d=0.25 , e=0.75

Substituting eq(21) in eq (1), we obtain

$$\underline{u}(x,\alpha) = \underline{f}(x,\alpha) + \int_{L(a)}^{L(d)} k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right) dt + \int_{L(d)}^{L(c)} k(x,t,\overline{k_1}(\overline{F}(t,\overline{u}(t,\alpha))dt + \int_{L(c)}^{L(e)} k(x,t,\underline{k_1}(t,\overline{F}(t,\overline{u}(t,\alpha))dt + \int_{L(e)}^{L(x)} k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right) dt$$

$$\overline{u}(x,\alpha) = \overline{f}(x,\alpha) + \int_{R(a)}^{R(d)} k(x,t,\overline{k_1}(\overline{F}(t,\overline{u}(t,\alpha)) + \int_{R(d)}^{R(c)} k(x,t,k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right) + \int_{R(c)}^{R(e)} k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right) dt + \int_{R(e)}^{R(x)} k(x,t,\underline{k_1}(t,\overline{F}(t,\overline{u}(t,\alpha))dt$$

We will use the all step in part 4 to find the solution of the system (2), . the numerical solution of the fuzzy function over the fuzzy interval using the LR-type representation of fuzzy interval. Some numerical examples are prepared to show the efficiency and accuracy of the methods

5. Numerical example

We will apply theHomotopy perturbation method to solve our problem and compared with the exact solution

Example 1

Consider the fuzzy nonlinear volterra integral equation

$\tilde{u}(x) = \tilde{f}(x) + \lambda \int_{\tilde{a}}^{\tilde{x}} k\left(x,t,\tilde{k}_1\left(t,\tilde{F}(t,\tilde{u}(t))\right)\right) dt$

Step 1

$$\underline{u}(x,\alpha) = \underline{f}(x,\alpha) + \int_{L(a)}^{L(c)} k(x,t,\underline{k_1\left(t,F(t,u(t,\alpha))\right)} + \int_{L(c)}^{L(x)} k(x,t,\overline{k_1\left(t,F(t,u(t,\alpha))\right)}$$

$$\overline{u}(x,\alpha) = \overline{f}(x,\alpha) + \int_{R(a)}^{R(c)} k(x,t,\overline{k_1\left(t,F(t,u(t,\alpha))\right)} + \int_{R(c)}^{R(x)} k(x,t,\underline{k_1\left(t,F(t,u(t,\alpha))\right)}$$

where $L(0) \le t \le L(c)$, $L(c) \le t \le L(x)$, where $R(0) \le t \le R(c)$, $R(c) \le t \le R(x) 0 \le \alpha \le 1$

than

$$\underline{u}(x,\alpha) = \underline{f}(x,\alpha) + \int_{L(a)}^{L(d)} k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right) dt + \int_{L(d)}^{L(c)} k(x,t,\overline{k_1}(\overline{F}(t,\overline{u}(t,\alpha))dt + \int_{L(c)}^{L(e)} k(x,t,\underline{k_1}(t,\overline{F}(t,\overline{u}(t,\alpha))dt + \int_{L(e)}^{L(x)} k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right) dt$$

$\overline{u}(x,\alpha) = \overline{f}(x,\alpha) + \int_{R(a)}^{R(d)} k(x,t,\overline{k_1}(\overline{F}(t,\overline{u}(t,\alpha)) + \int_{R(d)}^{R(c)} k(x,t,k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right) + \int_{R(c)}^{R(e)} k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right) dt$
$+\int_{R(e)}^{R(x)} k(x,t,\underline{k_1}(t,\overline{F}(t,\overline{u}(t,\alpha))dt$

where

$L(0) \le t \le L(d),\ L(\ d) \le t \le L(c),\ L(\ c) \le t \le L(e),\ L(e) \le t \le L(x),$

$R(0) \le t \le R(d),\quad R(\ d) \le t \le R(c),\quad R(\ c) \le t \le R(e),\ R(e) \le t \le R(x), 0 \le \alpha \le 1$

The exact solution $\underline{u}(x,\alpha) = x\alpha$, $\overline{u}(x,\alpha) = x(2\quad \alpha)$

The Kernal of our problem

$k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right)$ =xt^3($\underline{u}(t,\alpha)$)^2

$$k(x,t,\overline{k_1}(\overline{F}(t,\overline{u}(t,\alpha)) = xt\text{^}2(\overline{u}(t,\alpha))\text{^}2$$
$$k(x,t,\underline{k_1}(t,\overline{F}(t,\overline{u}(t,\alpha)) = xt\text{^}3(\overline{u}(t,\alpha))\text{^}2$$
$$k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right) = xt\text{^}2(\underline{u}(t,\alpha))\text{^}2$$

Remark 1:

Let the$\tilde{a} = \tilde{0}$ *and* $\tilde{t} = \tilde{1}$, $\tilde{a} = \tilde{0} = [L(0), R(0)]$ *and* $\tilde{x} = [L(x), R(x)]$

$\tilde{c} = [L(c), R(c)], \tilde{d} = [L(d), R(d)], \tilde{e} = [L(e), R(e)]$ (12)

Re-wirte the eq(12) with the fuzzy number formula

$\tilde{a} = \tilde{0} = [L(0), R(0)] = \left[0\quad \sqrt{1\quad \alpha}, 0 + \sqrt{1\quad \alpha}\right] = [\quad \sqrt{1\quad \alpha}, \sqrt{1\quad \alpha}]$

$\tilde{x} = [L(x), R(x)] = \left[x\quad \sqrt{1\quad \alpha}, x + \sqrt{1\quad \alpha}\right] = \left[x\quad \sqrt{1\quad \alpha}, x + \sqrt{1\quad \alpha}\right]$

$\tilde{c} = [L(c), R(c)] = \left[c\quad \sqrt{1\quad \alpha}, c + \sqrt{1\quad \alpha}\right] = \left[0.5\quad \sqrt{1\quad \alpha}, 0.5 + \sqrt{1\quad \alpha}\right]$

$\tilde{d} = [L(d), R(d)] = \left[d\quad \sqrt{1\quad \alpha}, d + \sqrt{1\quad \alpha}\right] = \left[0.25\quad \sqrt{1\quad \alpha}, 0.25 + \sqrt{1\quad \alpha}\right]$

$\tilde{e} = [L(e), R(e)] = \left[e\quad \sqrt{1\quad \alpha}, e + \sqrt{1\quad \alpha}\right] = \left[0.75\quad \sqrt{1\quad \alpha}, 0.75 + \sqrt{1\quad \alpha}\right]$

C=0.5 , d=0.25 , e=0.75

$$\underline{u}(x,\alpha) = \underline{f}(x,\alpha) + \int_{-\sqrt{1-\alpha}}^{0.25-\sqrt{1-\alpha}} k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right) dt + \int_{0.25-\sqrt{1-\alpha}}^{0.5-\sqrt{1-\alpha}} k(x,t,\overline{k_1}(\overline{F}(t,\overline{u}(t,\alpha))dt + \int_{0.5-\sqrt{1-\alpha}}^{0.75-\sqrt{1-\alpha}} k(x,t,\underline{k_1}(t,\overline{F}(t,\overline{u}(t,\alpha))dt + \int_{0.75-\sqrt{1-\alpha}}^{x-\sqrt{1-\alpha}} k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right) dt$$

$\overline{u}(x,\alpha) = \overline{f}(x,\alpha) + \int_{\sqrt{1-\alpha}}^{0.25+\sqrt{1-\alpha}} k(x,t,\overline{k_1}(\overline{F}(t,\overline{u}(t,\alpha)) + \int_{0.25+\sqrt{1-\alpha}}^{0.5+\sqrt{1-\alpha}} k(x,t,k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right) +$
$\int_{0.5+\sqrt{1-\alpha}}^{.075+\sqrt{1-\alpha}} k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u}(t,\alpha)\right) dt + \int_{0.75+\sqrt{1-\alpha}}^{x+\sqrt{1-\alpha}} k(x,t,\underline{k_1}(t,\overline{F}(t,\overline{u}(t,\alpha))dt$

Now we will find the lift and right for $\tilde{f}(x)$

$$\underline{f}(x,\alpha) = xr - \frac{1}{6}xr^2\left((0.25-\sqrt{1-r})^6 - (1-r)^3\right) - \frac{1}{5}x(2-r)^2\left((0.5-\sqrt{1-r})^5 - (0.25 - \sqrt{1-r})^5\right) - \frac{1}{6}x(2-r)^2\left((0.75-\sqrt{1-r})^6 - (0.5-\sqrt{1-r})^6\right) - \frac{1}{5}xr^2\left((x-\sqrt{1-r})^5 - (0.75-\sqrt{1-r})^5\right)$$

$\overline{f}(x,\alpha)$=

$$x(2-r)-\frac{1}{5}x(2-r)^2\left((0.25+\sqrt{1-r})^5-(1-r)^{5/2}\right)-\frac{1}{6}xr^2\left((0.5+\sqrt{1-r})^6-(0.25+\sqrt{1-r})^6\right)-\frac{1}{5}xr^2\left((0.75+\sqrt{1-r})^5-(0.5+\sqrt{1-r})^5\right)-\frac{1}{6}xr^2\left((x-\sqrt{1-r})^6-(0.75+\sqrt{1-r})^6\right)$$

Step 2

$$\mu_{\tilde{A}}(x)=\begin{cases} L\left(\frac{m \quad x}{\alpha}\right), & x\le m\\ R\left(\frac{x \quad m}{\beta}\right), & x\ge m\end{cases}$$

$$\mu_{\tilde{B}}(y)=\begin{matrix} L\left(\frac{\acute{m} \quad y}{\acute{\alpha}}\right), & y\le \acute{m}\\ R\left(\frac{x \quad \acute{m}}{\acute{\beta}}\right), & y\ge \acute{m}\end{matrix}$$

Then

$$\tilde{A}=(m,\alpha,\beta)_{LR}$$

and

$$\tilde{B}=(\acute{m},\acute{\alpha},\acute{\beta})_{LR}$$

Discretization of the above continuous fuzzy numbers $\tilde{A}$ *and* $\tilde{B}$ as follows:

Step 1 Let $\tilde{A}$ be continuous fuzzy number of the LR-type function and as follows, where

$$L(0)=L(\underline{u}(x,\alpha))^2=(x \quad \alpha)^2, \qquad L(x)=L\left(\underline{u}(x,\alpha)\right)^2=(x \quad (2 \quad \alpha))\hat{}2$$
$$R(0)=R(x \quad \alpha)\; R(x)=(x(2 \quad \alpha))$$

Given the spread number $\alpha=2,\beta=3$ and m=0(mean value of the zero fuzzy number). Then the membership function of he continuous zero fuzzy number can be represented as follows.

$$\mu_{\tilde{0}}(x)=\begin{matrix} \mathrm{L}((\frac{0 \quad x}{2}) \quad \alpha)\hat{}2 & x\in[\;\;2/\alpha,0]\\ 1 & x=2/\alpha\\ R((\frac{x \quad 0}{3}) \quad \alpha) & x\in[0,3/\alpha]\end{matrix}$$

now we calculate , $\tilde{B}$ be a continuous fuzzy number of the LR-type function.
Given the spread number $\alpha'=2$ *and* $\beta'=3$ and $\acute{m}$=1(mean value of the one fuzzy number) as follows:

$$\mu_{\tilde{1}}(t)=\begin{matrix} L((\frac{1 \quad t}{2}) \quad (2 \quad \alpha))\hat{}2, & t\in[\;\;1/(2 \quad \alpha),1]\\ 1 & t=1/(2 \quad \alpha)\\ R((\frac{t \quad 1}{3}) \quad (2 \quad \alpha)) & t\in[1,\frac{3}{2 \quad \alpha}+1]\end{matrix}$$

Thus, the membership functions of the continuous zero fuzzy number $\mu_{\tilde{0}}(x)$ and $\mu_{\tilde{1}}(y)$ are bounded, where

$$p_1=\quad 2/\alpha, p_2=3/\alpha$$
$$q_1=\quad 1/(2 \quad \alpha), q_2=\frac{3}{2 \quad \alpha}+1$$

step 3. A partition for the continuous zero fuzzy number can be as follows let $x_0=0$(mean value of zero fuzzy number). For the left hand side,

where$x_0\ge x,\; let\; N_1=2\; and\; N_2=3\,,t\;\; en$

$p = \frac{|x_0 - p_1|}{N_1} = 0.67\alpha$

Also, for the right hand side, where $x_0 \leq x,\ let\ N_2 = 3$

$p\ = \frac{|p_2 - x_0|}{N_2} = 1/\alpha$

so, for the left side

$$x_i = x_0 \quad i\ p\ ,\ i = 0,1,2$$

with the membership function

$$\mu(x_i) = ((\frac{x_0 \quad x_i}{2}) \quad \alpha)\ \hat{}\ 2$$

where $x_0 = 0\ , \alpha = 2\ and\ L =\ ((\frac{-x_i}{2}) \quad \alpha)\ \hat{}\ 2$. The following results of table 1 are obtained:

Table 1. Right speared of fuzzy number of lower bound

i	x_i	$\mu(x_i)$
0	0	0
1	-0.67 α	0.1122 α^4
2	-1.34 α	0.449 α^4

The right hand side

$$x_i = i\ p \quad i\ ,\ i = 1,2,3,$$

with the membership function

$$\mu(x_i) = R\left(\frac{x_i \quad x_0}{3}\right) \quad \alpha = (\frac{x_i}{3})\alpha$$

where $x_0 = 0\ , \beta = 3\ and\ R = (\frac{x_i}{3})\alpha$. The following results of table 2 are obtained:

Table 2. left speared of fuzzy number of lower bound

I	x_i	$\mu(x_i)$
1	1/ α	0.333
2	2/ α -2	0.667-0.667α
3	3/ α -3	1-α

Then, the approximate discrete fuzzy number for $\tilde{A}$ is

$\tilde{A}\ = \{(x_i, \mu(x_i))|$i=0,1,2,3,4,5,6}

Similarly, a partition for the continuous one fuzzy number can be as follows $t_0 = 1$ (mean value of one fuzzy number). For the left hand side, where

where$t_0 \geq t,\ let\ M_1 = 2, t\ \ en$

$q = \frac{|t_0 - q_1|}{M_1} =$ 1/2 -1/2(2- α)

Also, for the right hand side, where $t_0 \leq t,\ let\ M_2 = 3$

$q\ = \frac{|q_2 - t_0|}{M_2} =$ 1/(2- α)

so, for the left side

$$t_i = t_0 \quad i\ q\ ,\ i = 0,1,2$$

with the membership function

$$\mu(t_i) = ((\frac{t_0 \quad t_i}{2}) \quad (2 \quad \alpha))\ \hat{}\ 2$$

where $t_0 = 1, \alpha = 2$ and $L = ((\frac{t_0 - t_i}{2})\quad (2 \quad \alpha))$ ^2 . The following results of table 3 are obtained:

Table 3. Right speared of fuzzy number of upper bound

i	t_i	$\mu(t_i)$
0	1	O
1	1/2 1/4 2 α	1/4(1/2 + 1/4 2 α)^2(2 α)^2
2	2/4 2 α	1/4(1 2/4 2 α)^2(2 α)^2

The right hand side

$$t_i = it_0 \quad i \quad q \quad , \quad i = 1,2,3$$

with the membership function

$$\mu(t_i) = R\left(\frac{t_i \quad t_0}{3}\right)(2 \quad \alpha) = (\frac{t_i \quad t_0}{3})(2 \quad \alpha)$$

where $t_0 = 1$ $, \beta = 3$ and $R = \left(\frac{t_i - t_0}{3}\right)(2 \quad \alpha)$. The following results of table 4 are obtained:

Table 4. Left speared of fuzzy number of upper bound

i	t_i	$\mu(t_i)$
1	1-1/(2-α)	0.333
2	2-2/(2-α)	0.333(2-α) 0.667
3	3-3/(2-α)	0.667(2-α) 1

$$\tilde{B} \quad = \{(t_i, \mu_{\tilde{B}}(t_i)|i = 0,1, \ldots\ldots,6\}$$

={(1,0),(1/2-1/4-2α, 1/4(1/2+1/4-2α)^2(2-α)^2),(2/4-2α, 1/4(1-2/4-2α)^2(2- α)^2),(1+1/(2-α),0.333),(2+2/(2-α), 0.333(2-α) 0.667),(3+3/(2-α), 0.667(2-α) 1)}

Step 2. Define approximately

$$\tilde{A} \times \tilde{B} \quad \{(x_i, y_j)|x_i \in \tilde{A}\ , y_j \in \tilde{B}\ \ , i = 0,1,2,3,4,5,6,, j = 0,1,2,3,4,5,6,\}$$

And then evaluate the integration on $\tilde{A} \times \tilde{B}$, where

$$I_{ij} = \int_{x_i}^{y_j} (f_k(y_j)) \quad dy_j$$
$$\mu(I_{ij}) = Min\{(\mu_{\tilde{A}}(x_i), \mu_{\tilde{B}}(y_j)\}$$

finally we use the HPM to find the solution

substitution or all $x = x_i$ and $t = t_j$

we have 36 values for lift sider and 36 for right side of the the equation with the Minimum membership function

$u_{k1}(x) = \{(f(x_i)$+I,$I_{ij}, MIn\ (\mu(f(x_i)$+$I, I_{ij}))\}$

and finally we find the $u_{kn}(x)$ we have 36 values for lif and 36 for right of the equation with membership function

step 4. Check if $I_{i1,j1} = I_{12,j2}, i1 \neq j2\ , i2 \neq j2$, then

$$\mu(I_{i,j}) = Sup\{\mu(I_{i1,j1}), \mu(I_{12,j2})\}$$

we have the number of crisp nonlinear function over fuzzy life side integration are reduced from 36 to 9 , and for right side integration are reduce from 36 to 9 step 5. Finally, we have the total crisp nonlinear function over fuzzy integration

$$I(\tilde{A},\tilde{B}) = I(\tilde{0},\tilde{1}) = \left\{\left(I_{ij},\mu(I_{i,j})\right)\right) | i = 0,1,\dots,I \in N ; j = 0,1,\dots,J \in N\}$$

The lower integral equation

$$(\tilde{A},\tilde{B}) = \left\{\left[\ 1.34\alpha, \frac{2}{4\quad 2\alpha}\right], \left[\ 1.34\alpha, \frac{1}{2}\quad \frac{1}{4\quad 2\alpha}\right], [\ 1.34\alpha, 1]\right\},$$
$$\left\{\left[\ 0.67\alpha, \frac{2}{4\quad 2\alpha}\right], \left[\ 0.67\alpha, \frac{1}{2}\quad \frac{1}{4\quad 2\alpha}\right], [\ 0.67\alpha, 1]\right\},$$
$$\left\{\left[\frac{0,2}{4\quad 2\alpha}\right], \left[\frac{0,1}{2}\quad \frac{1}{4\quad 2\alpha}\right], [0,1]\right\}$$

The upper integral equation and using the Remark 2

$$(\tilde{A},\tilde{B}) = \left\{\left[1\quad \frac{1}{2\quad \alpha}, \frac{1}{\alpha}\right], \left[\left[1\quad \frac{1}{2\quad \alpha}, \frac{2}{\alpha}\quad 2\right]\right], \left[1\quad \frac{1}{2\quad \alpha}, \frac{3}{\alpha}\quad 3\right]\right\},$$
$$\left\{\left[2\quad \frac{2}{2\quad \alpha}, \frac{1}{\alpha}\right], \left[\left[2\quad \frac{2}{2\quad \alpha}, \frac{2}{\alpha}\quad 2\right]\right], \left[2\quad \frac{2}{2\quad \alpha}, \frac{3}{\alpha}\quad 3\right]\right\},$$
$$\left\{\left[3\quad \frac{3}{2\quad \alpha}, \frac{1}{\alpha}\right], \left[\left[3\quad \frac{3}{2\quad \alpha}, \frac{2}{\alpha}\quad 2\right]\right], \left[3\quad \frac{3}{2\quad \alpha}, \frac{3}{\alpha}\quad 3\right]\right\}$$

we will have 81 integral of the fuzzy intervals

Now we reduce the 9 integral for lower and 9 integral for upper

Now we will applied the HPM to solve our formula

$$\underline{u}_0(x,\alpha) = \underline{f}(x,\alpha) = \begin{array}{l} x\,r - \frac{1}{6}x\,r^2\left((0.25 - \sqrt{1-r})^6 - (1-r)^3\right) - \frac{1}{5}x\,(2-r)^2\left((0.5 - \sqrt{1-r})^5 - (0.25\right. \\ \left. - \sqrt{1-r})^5\right) - \frac{1}{6}x\,(2-r)^2\left((0.75 - \sqrt{1-r})^6 - (0.5 - \sqrt{1-r})^6\right) \\ - \frac{1}{5}x\,r^2\left((x - \sqrt{1-r})^5 - (0.75 - \sqrt{1-r})^5\right) \end{array}$$

$$\overline{u}_0(x,\alpha) = \overline{f}(x,\alpha) = \begin{array}{l} x\,(2-r) - \frac{1}{5}x\,(2-r)^2\left((0.25 + \sqrt{1-r})^5 - (1-r)^{5/2}\right) - \frac{1}{6}x\,r^2\left((0.5 + \sqrt{1-r})^6\right. \\ \left. - (0.25 + \sqrt{1-r})^6\right) - \frac{1}{5}x\,r^2\left((0.75 + \sqrt{1-r})^5 - (0.5 + \sqrt{1-r})^5\right) \\ - \frac{1}{6}x\,r^2\left((x - \sqrt{1-r})^6 - (0.75 + \sqrt{1-r})^6\right) \end{array}$$

$$\underline{u}_1(x,\alpha) = \int_{-\sqrt{1-\alpha}}^{0.25-\sqrt{1-\alpha}} k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u_0}(t,\alpha)\right)dt + \int_{0.25-\sqrt{1-\alpha}}^{0.5-\sqrt{1-\alpha}} k(x,t,\overline{k_1}(\overline{F}\left(t,\overline{u_0}(t,\alpha)\right)dt$$
$$+ \int_{0.5-\sqrt{1-\alpha}}^{0.75-\sqrt{1-\alpha}} k(x,t,\underline{k_1}(t,\overline{F}\left(t,\overline{u_0}(t,\alpha)\right)dt + \int_{0.75-\sqrt{1-\alpha}}^{x-\sqrt{1-\alpha}} k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u}_0(t,\alpha)\right)dt$$

$$\overline{u}_1(x,\alpha) = \int_{R(\alpha)}^{R(d)} k(x,t,\overline{k_1}(\overline{F}\left(t,\overline{u_0}(t,\alpha)\right) + \int_{R(d)}^{R(c)} k(x,t,k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u}_0(t,\alpha)\right) + \int_{R(c)}^{R(e)} k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u}_0(t,\alpha)\right)dt$$
$$+\int_{R(e)}^{R(x)} k(x,t,\underline{k_1}(t,\overline{F}\left(t,\overline{u_0}(t,\alpha)\right)dt$$

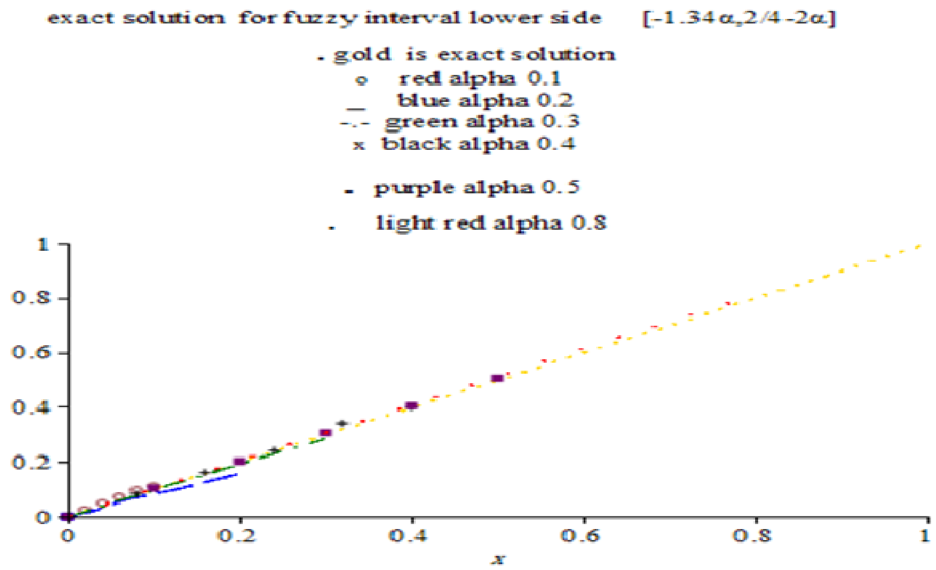
And now we will find the value of the integration for the lower partition for interval by using HPM
$\underline{u}_1(x,\alpha)$in interval $[\ 1.34\alpha, \frac{2}{4-2\alpha}]$

$$\int_{-1.34\alpha}^{0.25-\sqrt{1-\alpha}} k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u}_0(t,\alpha)\right)dt + \int_{0.25-\sqrt{1-\alpha}}^{0.5-\sqrt{1-\alpha}} k(x,t,\overline{k_1}(\overline{F}\left(t,\overline{u_0}(t,\alpha)\right)dt + \int_{0.5-\sqrt{1-\alpha}}^{0.75-\sqrt{1-\alpha}} k(x,t,\underline{k_1}(t,\overline{F}\left(t,\overline{u_0}(t,\alpha)\right)dt$$
$$+ \int_{0.75-\sqrt{1-\alpha}}^{\frac{2}{4-2\alpha}} k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u}_0(t,\alpha)\right)dt$$

Table 1 . For lower bound interval(-1.34α,2/4-2α) for $\underline{u_1}(x,\alpha)$

X	Exact $\alpha = 0.1$	HPM α=0.1	Exact $\alpha = 0.2$	HPM $\alpha = 0.2$	Exact α=0.3	HPM α=0.3	Exact $\alpha = 0.4$	HPM $\alpha = 0.4$	Exact $\alpha = 0.5$	HPM $\alpha = 0.5$	Exact $\alpha = 0.8$	HPM $\alpha = 0.8$
0	0.0000	0.00000	0.0000	0.000000	0.000	0.000000000	0.0000	0.000000000	0.000	0.0000	0.0000	0.0000000000
0.2	0.0200	0.0249	0.0400	0.316674	0.060	0.057385673	0.0800	0.080645581	0.100	0.1015361920	0.1600	0.1634133051
0.4	0.0400	0.0483	0.0800	0.062919	0.120	0.114335743	0.1600	0.160571267	0.200	0.2024512711	0.3200	0.3267774894
0.6	0.0600	0.0719	0.1200	0.094249	0.180	0.171340120	0.2400	0.240727321	0.300	0.3035945575	0.4800	0.4901595691
0.8	0.0800	0.0958	0.1600	0.125652	0.240	0.228442746	0.3200	0.320965546	0.400	0.4047916926	0.6400	0.6529732694
1	0.1000	0.1198	0.2000	0.157064	0.300	0.285551307	0.4000	0.401187925	0.500	0.5058808041	0.8000	0.8100762465

Fig 1.Compared between the exact solution and HPM



$$\int_{-1.34\alpha}^{0.25-\sqrt{1-\alpha}} k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u_0}(t,\alpha)\right)dt + \int_{0.25-\sqrt{1-\alpha}}^{0.5-\sqrt{1-\alpha}} k(x,t,\overline{k_1}(\overline{F}\left(t,\overline{u_0}(t,\alpha)\right)dt + \int_{0.5-\sqrt{1-\alpha}}^{0.75-\sqrt{1-\alpha}} k(x,t,\underline{k_1}(t,\overline{F}\left(t,\overline{u_0}(t,\alpha)\right)dt$$
$$+ \int_{0.75-\sqrt{1-\alpha}}^{\frac{1}{2}-\frac{1}{4-2\alpha}} k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u_0}(t,\alpha)\right)dt$$

Table 2. For lower bound interval [-1.34α,1/2- 1/4-2α] for $\underline{u_1}(x,\alpha)$

X	Exact $\alpha = 0.1$	HPM α=0.1	Exact $\alpha = 0.2$	HPM $\alpha = 0.2$	Exact α=0.3	HPM α=0.3	Exact $\alpha = 0.4$	HPM $\alpha = 0.4$	Exact 0.5	HPM 0.5	Exact $\alpha = 0.8$	HPM $\alpha = 0.8$
0	0.0000	0.0000	0.0000	0.000000	0.000	0.000000	0.0000	0.0000000	0.000	0.000	0.0000	0.00000
0.2	0.0200	0.024890	0.0400	0.03162591280	0.060	0.05727200217	0.0800	0.07995464738	0.100	0.100232548	0.1600	0.127734392
0.4	0.0400	0.048335	0.0800	0.06283785628	0.120	0.1139134510	0.1600	0.1592089707	0.200	0.199860092	0.3200	0.255440714
0.6	0.0600	0.071977	0.1200	0.09412721372	0.180	0.1707079694	0.2400	0.2386758324	0.300	0.299709918	0.4800	0.381572623
0.8	0.0800	0.095889	0.1600	0.1254889212	0.240	0.2275999625	0.3200	0.3182303024	0.400	0.399612396	0.6400	0.510548830
1	0.1000	0.119860	0.2000	0.1568609859	0.300	0.2844978439	0.4000	0.3977691990	0.500	0.499409237	0.8000	0.634664580

$\underline{u_1}(x,\alpha)$in interval [1.34α, 1]

$$\int_{-1.34\alpha}^{0.25-\sqrt{1-\alpha}} k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u_0}(t,\alpha)\right)dt + \int_{0.25-\sqrt{1-\alpha}}^{0.5-\sqrt{1-\alpha}} k(x,t,\overline{k_1}(\overline{F}\left(t,\overline{u_0}(t,\alpha)\right)dt + \int_{0.5-\sqrt{1-\alpha}}^{0.75-\sqrt{1-\alpha}} k(x,t,\underline{k_1}(t,\overline{F}\left(t,\overline{u_0}(t,\alpha)\right)dt$$
$$+ \int_{0.75-\sqrt{1-\alpha}}^{1} k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u_0}(t,\alpha)\right)dt$$

Table (3) for lower bound interval [-1.34α,1]

Fig. 2. Compared between the exact solution and HPM

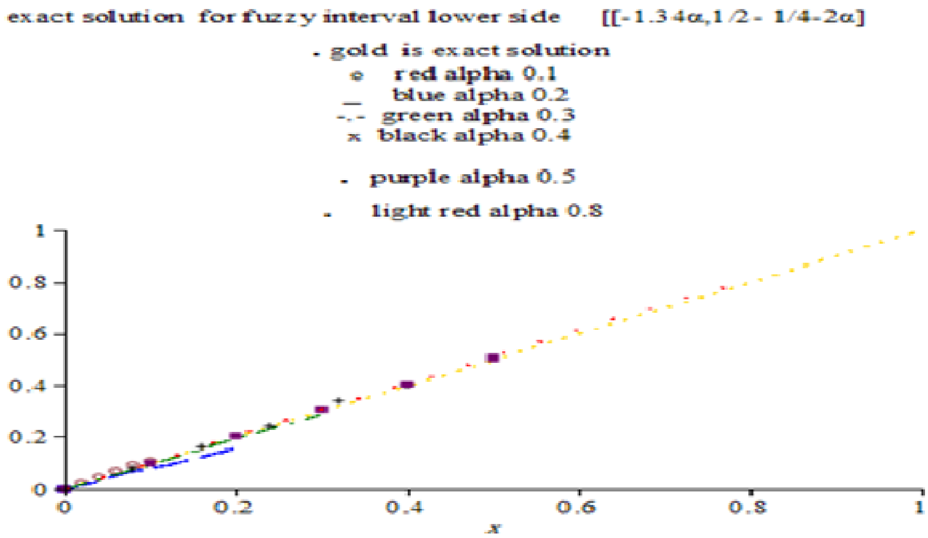
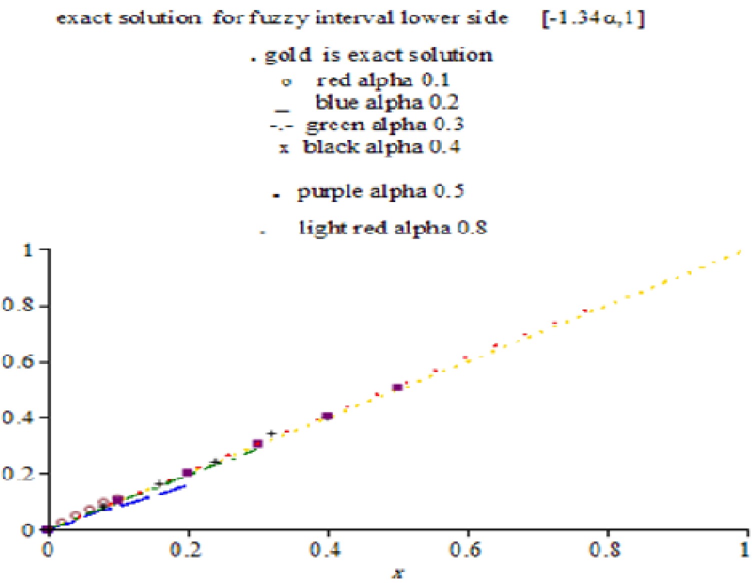


Table 3. For lower bound interval (-1.34α,1)

x	Exact $\alpha = \mathbf{0.1}$	HPM α=0.1	Exact $\alpha = \mathbf{0.2}$	HPM $\alpha = \mathbf{0.2}$	Exact α=0.3	HPM α=0.3	Exact $\alpha = \mathbf{0.4}$	HPM $\alpha = \mathbf{0.4}$	Exact 0.5	HPM 0.5	Exact $\alpha = \mathbf{0.8}$	HPM $\alpha = \mathbf{0.8}$
0	0.0000	0.000000	0.0000	0.000000000	0.000	0.0000000000	0.0000	0.0000000000	0.000	0.0000000	0.0000	0.00000000
0.2	0.0200	0.024890	0.0400	0.032414231	0.060	0.0601750779	0.0800	0.0859703147	0.100	0.11014050	0.1600	0.15339961
0.4	0.0400	0.048336	0.0800	0.064390878	0.120	0.1196509785	0.1600	0.1711308488	0.200	0.21955355	0.3200	0.30675603
0.6	0.0600	0.071979	0.1200	0.096449444	0.180	0.1792967655	0.2400	0.2565372716	0.300	0.32923393	0.4800	0.46012815
0.8	0.0800	0.095892	0.1600	0.128584443	0.240	0.2390505471	0.3200	0.3426449136	0.400	0.43897758	0.6400	0.61300023
1	0.1000	0.119863	0.2000	0.160730379	0.300	0.2988108506	0.4000	0.4275345903	0.500	0.54859441	0.8000	0.76084497

Fig. 3. Compared between the exact solution and HPM



$\underline{u}_1(x,\alpha)$in interval [$0.67\alpha, \frac{2}{(4-2\ r)}$]

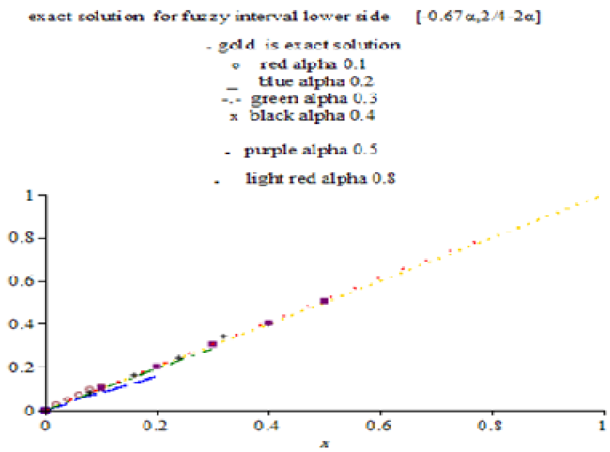
$$\int_{-0.67\alpha}^{0.25-\sqrt{1-\alpha}} k(x,t,\underline{k}_1(t,\underline{F}\left(t,\underline{u}_0(t,\alpha)\right)dt + \int_{0.25-\sqrt{1-\alpha}}^{0.5-\sqrt{1-\alpha}} k(x,t,\overline{k_1}(\overline{F}\left(t,\overline{u_0}(t,\alpha)\right)dt + \int_{0.5-\sqrt{1-\alpha}}^{0.75-\sqrt{1-\alpha}} k(x,t,\underline{k}_1(t,\overline{F}\left(t,\overline{u_0}(t,\alpha)\right)dt$$

$$+\int_{0.75-\sqrt{1-\alpha}}^{\frac{2}{(4-2\ r)}} k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u}_0(t,\alpha)\right)dt$$

Table 4. For lower bound interval (-0.67α,2/4-2α)

X	Exact $\alpha = 0.1$	HPM α=0.1	Exact $\alpha = 0.2$	HPM $\alpha = 0.2$	Exact α=0.3	HPM α=0.3	Exact $\alpha = 0.4$	HPM $\alpha = 0.4$	Exact 0.5	HPM 0.5	Exact $\alpha = 0.8$	HPM $\alpha = 0.8$
0	0.0000	0.00000000	0.0000	0.0000000000	0.000	0.0000000000	0.0000	0.0000000000	0.000	0.00000000	0.0000	0.000000000
0.2	0.0200	0.02489036	0.0400	0.0316674671	0.060	0.0574855000	0.0800	0.0806441300	0.100	0.10227149	0.1600	0.1700003675
0.4	0.0400	0.04833511	0.0800	0.0629197203	0.120	0.1143354403	0.1600	0.1605753899	0.200	0.20912795	0.3200	0.3399477278
0.6	0.0600	0.71977659	0.1200	0.0942496248	0.180	0.1713396667	0.2400	0.2407230109	0.300	0.30578564	0.4800	0.5099143994
0.8	0.0800	0.09588984	0.1600	0.1256520946	0.240	0.2284421450	0.3200	0.3209598002	0.400	0.40771332	0.6400	0.6792677553
1	0.1000	0.11986084	0.2000	0.1570649522	0.300	0.2855505512	0.4000	0.4011080742	0.500	0.50953101	0.8000	0.8424608540

Fig. 4. Compared between exact solution and HPM



$\underline{u}_1(x,\alpha)$in interval [$0.67\alpha, \frac{1}{2}$ $\frac{1}{4-2\alpha}$]

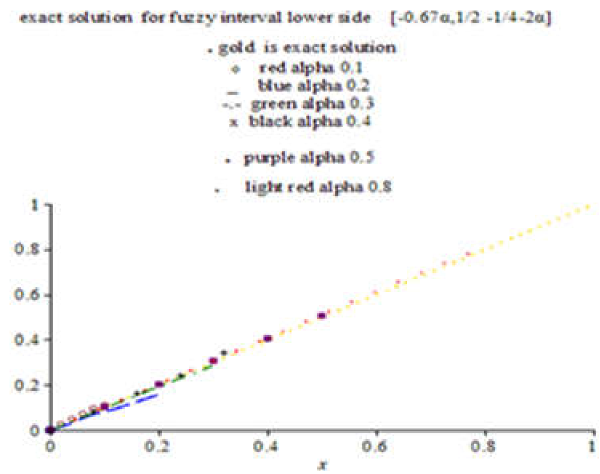
$$\int_{-0.67\alpha}^{0.25-\sqrt{1-\alpha}} k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u_0}(t,\alpha)\right)dt + \int_{0.25-\sqrt{1-\alpha}}^{0.5-\sqrt{1-\alpha}} k(x,t,\overline{k_1}(\overline{F}\left(t,\overline{u_0}(t,\alpha)\right)dt + \int_{0.5-\sqrt{1-\alpha}}^{0.75-\sqrt{1-\alpha}} k(x,t,\underline{k_1}(t,\overline{F}\left(t,\overline{u_0}(t,\alpha)\right)dt$$

$$+ \int_{0.75-\sqrt{1-\alpha}}^{\frac{1}{2}-\frac{1}{4-2\alpha}} k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u}_0(t,\alpha)\right)dt$$

Table 5. For lower bound interval (-0.67α,1/2 -1/4-2α)

X	Exact $\alpha = 0.1$	HPM α=0.1	Exact $\alpha = 0.2$	HPM $\alpha = 0.2$	Exact α=0.3	HPM α=0.3	Exact $\alpha = 0.4$	HPM $\alpha = 0.4$	Exact 0.5	HPM 0.5	Exact $\alpha = 0.8$	HPM $\alpha = 0.8$
0	0.0000	0.00000000	0.0000	0.0000000000	0.000	0.000000000	0.0000	0.000000000	0.000	0.00000000	0.0000	0.0000000000
0.2	0.0200	0.02489035	0.0400	0.3162615253	0.060	0.0572820565	0.0800	0.0800716923	0.100	0.10096785	0.1600	0.1596861526
0.4	0.0400	0.04833508	0.0800	0.0628383285	0.120	0.1139333220	0.1600	0.1594409309	0.200	0.20132161	0.3200	0.3193253834
0.6	0.0600	0.07197759	0.1200	0.0941279199	0.180	0.1707377154	0.2400	0.2390233567	0.300	0.30190100	0.4800	0.4789817086
0.8	0.0800	0.09588974	0.1600	0.1254898626	0.240	0.2276396199	0.3200	0.3186936556	0.400	0.40253383	0.6400	0.6380950806
1	0.1000	0.11986007	0.2000	0.1568621626	0.300	0.2845474149	0.4000	0.3983483347	0.500	0.50305945	0.8000	0.7917520887

Fig.5. Compared between exact and HPM

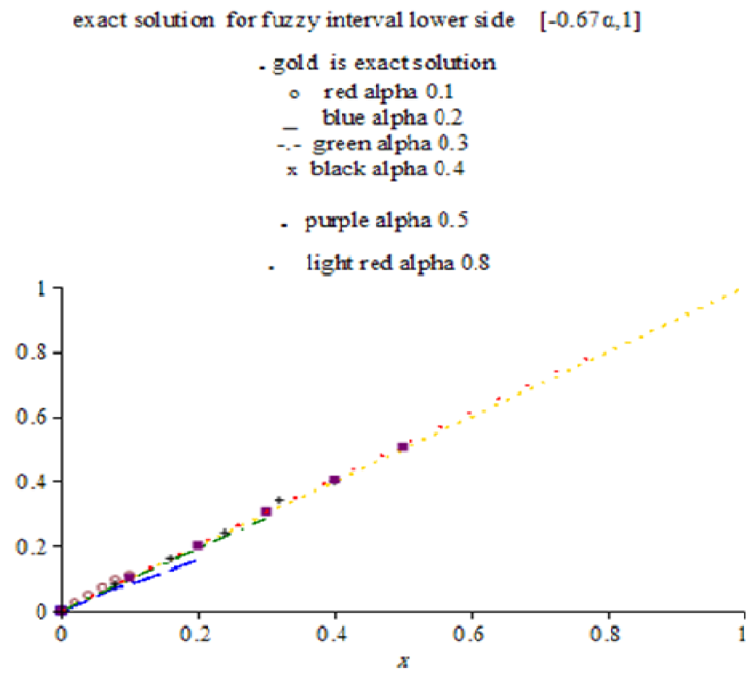


$\int_{-0.67\alpha}^{0.25-\sqrt{1-\alpha}} k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u_0}(t,\alpha)\right)dt + \int_{0.25-\sqrt{1-\alpha}}^{0.5-\sqrt{1-\alpha}} k(x,t,\overline{k_1}(\overline{F}\left(t,\overline{u_0}(t,\alpha)\right)dt + \int_{0.5-\sqrt{1-\alpha}}^{0.75-\sqrt{1-\alpha}} k(x,t,\underline{k_1}(t,\overline{F}\left(t,\overline{u_0}(t,\alpha)\right)dt +$
$\int_{0.75-\sqrt{1-\alpha}}^{1} k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u}_0(t,\alpha)\right)dt$

Table 6. For lower bound interval (-0.67α,1)

x	Exact $\alpha = 0.1$	HPM α=0.1	Exact $\alpha = 0.2$	HPM $\alpha = 0.2$	Exact α=0.3	HPM α=0.3	Exact $\alpha = 0.4$	HPM $\alpha = 0.4$	Exact 0.5	HPM 0.5	Exact $\alpha = 0.8$	HPM $\alpha = 0.8$
0	0.0000	0.00000000	0.0000	0.0000000000	0.000	00000000000	0.0000	0.0000000000	0.000	0.00000000	0.0000	0.0000000000
0.2	0.0200	0.02489056	0.0400	0.0324144705	0.060	0.0601851322	0.0800	0.0860873596	0.100	0.11087580	0.1600	0.1853513732
0.4	0.0400	0.04833603	0.0800	0.0643913516	0.120	0.1196708496	0.1600	0.1713628090	0.200	0.22101507	0.3200	0.3706406822
0.6	0.0600	0.07197926	0.1200	0.0964501500	0.180	0.1793265115	0.2400	0.2568847958	0.300	0.33142502	0.4800	0.5595260290
0.8	0.0800	0.09589200	0.1600	0.1285853843	0.240	0.2390920440	0.3200	0.3425082669	0.400	0.44189901	0.6400	0.7405464832
1	0.1000	0.11986355	0.2000	0.1607315556	0.300	0.2988604215	0.4000	0.4281137260	0.500	0.55224462	0.8000	0.9179324782

Fig.6. Compared between exact solution and HPM



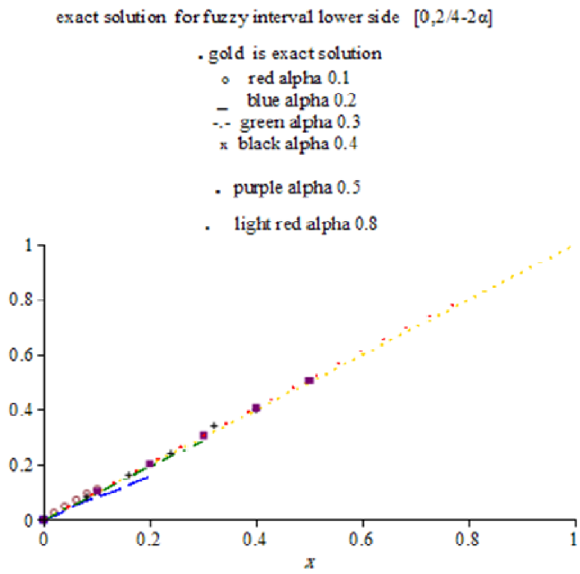
$\underline{u}_1(x,\alpha)$in interval $[0,2/(4 \quad 2\alpha)]$

$$\int_0^{0.25-\sqrt{1-\alpha}} k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u_0}(t,\alpha)\right)dt + \int_{0.25-\sqrt{1-\alpha}}^{0.5-\sqrt{1-\alpha}} k(x,t,\overline{k_1}(\overline{F}\left(t,\overline{u_0}(t,\alpha)\right)dt + \int_{0.5-\sqrt{1-\alpha}}^{0.75-\sqrt{1-\alpha}} k(x,t,\underline{k_1}(t,\overline{F}\left(t,\overline{u_0}(t,\alpha)\right)dt + \int_{0.75-\sqrt{1-\alpha}}^{2/(4-2\alpha)} k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u}_0(t,\alpha)\right)dt$$

Table 7. For lower bound interval (0,2/4-2α)

X	Exact $\alpha = 0.1$	HPM α=0.1	Exact $\alpha = 0.2$	HPM $\alpha = 0.2$	Exact α=0.3	HPM α=0.3	Exact $\alpha = 0.4$	HPM $\alpha = 0.4$	Exact 0.5	HPM 0.5	Exact $\alpha = 0.8$	HPM $\alpha = 0.8$
0	0.0000	0.00000000	0.0000	0.0000000000	0.000	0.000000000	0.0000	0.0000000000	0.000	0.00000000	0.0000	0.000000000
0.2	0.0200	0.02489036	0.0400	0.0316674710	0.060	0.057485680	0.0800	0.0806459875	0.100	0.10228317	0.1600	0.1705075383
0.4	0.0400	0.04833511	0.0800	0.0629197278	0.120	0.114335756	0.1600	0.1605790718	0.200	0.20393599	0.3200	0.3409617701
0.6	0.0600	0.07197766	0.1200	0.0942496360	0.180	0.171340139	0.2400	0.2407285270	0.300	0.30582042	0.4800	0.5114354224
0.8	0.0800	0.09588983	0.1600	0.1256521096	0.240	0.228442770	0.3200	0.3209671551	0.400	0.40775970	0.6400	0.6812922989
1	0.1000	0.11986084	0.2000	0.1570649709	0.300	0.285551330	0.4000	0.4011899347	0.500	0.50958895	0.8000	0.8449543065

Fig. 7. Compared between exact and HPM



$\underline{u}_1(x,\alpha)$in interval $[0,\frac{1}{2} \quad \frac{1}{4-2\alpha}]$

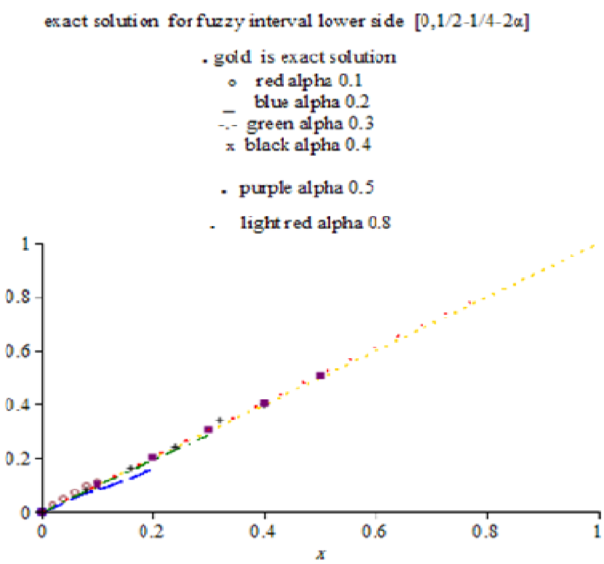
$$\int_0^{0.25-\sqrt{1-\alpha}} k(x,t,\underline{k}_1(t,\underline{F}\left(t,\underline{u}_0(t,\alpha)\right)dt + \int_{0.25-\sqrt{1-\alpha}}^{0.5-\sqrt{1-\alpha}} k(x,t,\overline{k_1}(\overline{F}\left(t,\overline{u_0}(t,\alpha)\right)dt + \int_{0.5-\sqrt{1-\alpha}}^{0.75-\sqrt{1-\alpha}} k(x,t,\underline{k}_1(t,\overline{F}\left(t,\overline{u_0}(t,\alpha)\right)dt$$

$$+\int_{0.75-\sqrt{1-\alpha}}^{\frac{1}{2}-\frac{1}{4-2\alpha}} k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u}_0(t,\alpha)\right)dt$$

Table 8. For lower bound interval (0,1/2-1/4-2α)

x	Exact $\alpha = 0.1$	HPM α=0.1	Exact $\alpha = 0.2$	HPM $\alpha = 0.2$	Exact α=0.3	HPM α=0.3	Exact $\alpha = 0.4$	HPM $\alpha = 0.4$	Exact 0.5	HPM 0.5	Exact $\alpha = 0.8$	HPM $\alpha = 0.8$
0	0.0000	0.00000000	0.0000	0.0000000000	0.000	0.0000000000	0.0000	0.0000000000	0.000	0.00000000	0.0000	0.0000000000
0.2	0.0200	0.02489035	0.0400	0.0316261563	0.060	0.0572822161	0.0800	0.0800735501	0.100	0.10097952	0.1600	0.1601933234
0.4	0.0400	0.04835081	0.0800	0.0628383360	0.120	0.1139336374	0.1600	0.1594446128	0.200	0.20134481	0.3200	0.3203394258
0.6	0.0600	0.07197759	0.1200	0.0941279312	0.180	0.1707381875	0.2400	0.2390288730	0.300	0.30193578	0.4800	0.4805027314
0.8	0.0800	0.09588974	0.1600	0.1254898775	0.240	0.2276402494	0.3200	0.3187010105	0.400	0.40258020	0.6400	0.6401196242
1	0.1000	0.11986072	0.2000	0.1568621813	0.300	0.2845482017	0.4000	0.3983575273	0.500	0.50311739	0.8000	0.7942455412

Fig. 8. Compared between exact and HPM



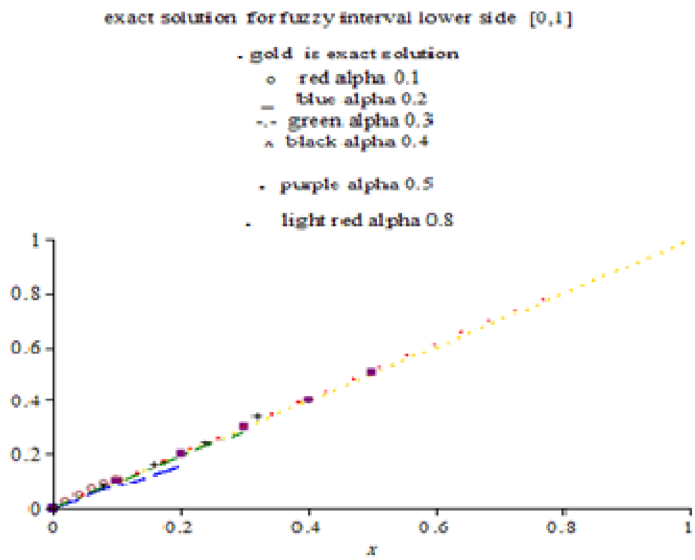
$\underline{u}_1(x,\alpha)$in interval [0,1]

$$\int_0^{0.25-\sqrt{1-\alpha}} k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u_0}(t,\alpha)\right)dt + \int_{0.25-\sqrt{1-\alpha}}^{0.5-\sqrt{1-\alpha}} k(x,t,\overline{k_1}(\overline{F}\left(t,\overline{u_0}(t,\alpha)\right)dt + \int_{0.5-\sqrt{1-\alpha}}^{0.75-\sqrt{1-\alpha}} k(x,t,\underline{k_1}(t,\overline{F}\left(t,\overline{u_0}(t,\alpha)\right)dt + \int_{0.75-\sqrt{1-\alpha}}^{1} k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u_0}(t,\alpha)\right)dt$$

Table 9. For lower bound interval (0,1)

X	Exact $\alpha = 0.1$	HPM α=0.1	Exact $\alpha = 0.2$	HPM $\alpha = 0.2$	Exact α=0.3	HPM α=0.3	Exact $\alpha = 0.4$	HPM $\alpha = 0.4$	Exact 0.5	HPM 0.5	Exact $\alpha = 0.8$	HPM $\alpha = 0.8$
0	0.0000	0.0000000	0.0000	0.0000000000	0.000	0.0000000000	0.0000	0.000000000	0.000	0.00000000	0.0000	0.0000000000
0.2	0.0200	0.02490563	0.0400	0.0324144743	0.060	0.0601852918	0.0800	0.0860892175	0.100	0.11088748	0.1600	0.1858585440
0.4	0.0400	0.04833602	0.0800	0.0643913591	0.120	0.1196711650	0.1600	0.1713664909	0.200	0.22103827	0.3200	0.3716547246
0.6	0.0600	0.07197925	0.1200	0.0945016122	0.180	0.1793269837	0.2400	0.2568903121	0.300	0.33145979	0.4800	0.5574736254
0.8	0.0800	0.09589200	0.1600	0.1285853992	0.240	0.2390908334	0.3200	0.3425156217	0.400	0.44194539	0.6400	0.7425710269
1	0.1000	0.11986354	0.2000	0.1067315743	0.300	0.2988612083	0.4000	0.4281229186	0.500	0.55230256	0.8000	0.9204259307

Fig.9. Compared between exact and HPM



Now we will find the upper side

$\overline{u}_1(x,\alpha) = \int_{R(a)}^{R(d)} k(x,t,\overline{k_1}(\overline{F}\left(t,\overline{u_0}(t,\alpha)\right) + \int_{R(d)}^{R(c)} k(x,t,k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u}_0(t,\alpha)\right) + \int_{R(c)}^{R(e)} k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u}_0(t,\alpha)\right)dt$
$+\int_{R(e)}^{R(x)} k(x,t,\underline{k_1}(t,\overline{F}\left(t,\overline{u_0}(t,\alpha)\right)dt$

And now we will find the value of the integration for the upper partition for interval by using HPM

$\overline{u}_1(x,\alpha)$ =in interval $\left[1 \quad \frac{1}{2-\alpha} \quad \alpha, \frac{1}{\alpha}\right]$

$\int_{1-1/2}^{-0.25+\sqrt{1-\alpha}} k(x,t,\overline{k_1}(\overline{F}\left(t,\overline{u_0}(t,\alpha)\right) + \int_{-0.25+\sqrt{1-\alpha}}^{-0.5+\sqrt{1-\alpha}} k(x,t,k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u}_0(t,\alpha)\right) + \int_{-0.5+\sqrt{1-\alpha}}^{-0.75+\sqrt{1-\alpha}} k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u}_0(t,\alpha)\right)dt$
$+\int_{-0.75+\sqrt{1-\alpha}}^{\frac{1}{\alpha}} k(x,t,\underline{k_1}(t,\overline{F}\left(t,\overline{u_0}(t,\alpha)\right)dt$

$$\overline{u}_0(x,\alpha) = \overline{f}(x,\alpha) = x(2-r) - \frac{1}{5}x(2-r)^2\left((0.25+\sqrt{1-r})^5 - (1-r)^{5/2}\right) - \frac{1}{6}xr^2\left((0.5+\sqrt{1-r})^6 - (0.25+\sqrt{1-r})^6\right) - \frac{1}{5}xr^2\left((0.75+\sqrt{1-r})^5 - (0.5+\sqrt{1-r})^5\right) - \frac{1}{6}xr^2\left((x-\sqrt{1-r})^6 - (0.75+\sqrt{1-r})^6\right)$$

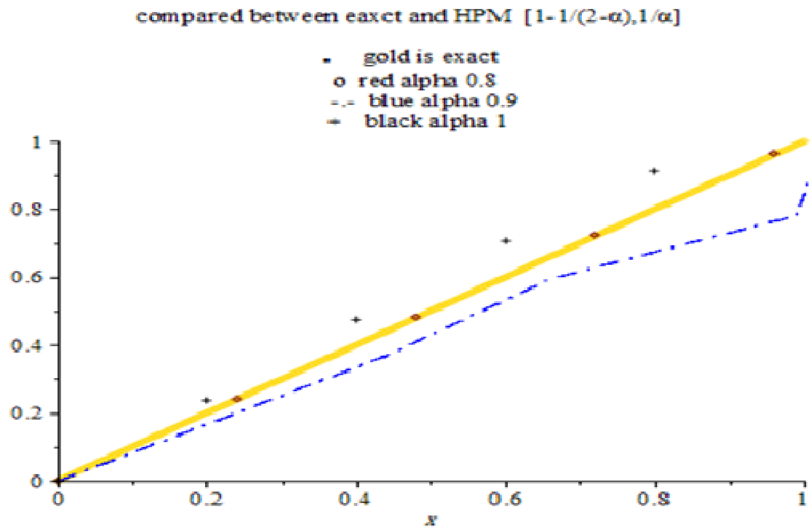
$\overline{u}_1(x,\alpha) =$

$$\overline{u}(x,\alpha) = \overline{u}_0(x,\alpha) + p^1\overline{u}_1(x,\alpha) + \quad \dots$$

Table 10. For upper bound interval $(1 \quad \frac{1}{2-\alpha}, \frac{1}{\alpha})$

x	Exact $\alpha = 0.8$	HPM $\alpha = 0.8$	Exact $\alpha = 0.9$	HPM $\alpha = 0.9$	Exact $\alpha = 1$	HPM $\alpha = 1$
0	0.00000	0.0000000000	0.000000	0.0000000000	0.00000	0.0000000000
0.2	0.24000	0.2411826921	0.220000	0.1832759126	0.20000	0.2370155469
0.4	0.48000	0.4823872965	0.440000	0.3665519962	0.40000	0.4737623581
0.6	0.72000	0.7235791148	0.660000	0.5947648550	0.60000	0.7063880020
0.8	0.96000	0.9644044580	0.880000	0.7810359563	0.80000	0.9131222690
1	1.20000	1.1991204420	1.100000	1.4260855860	1.00000	1.0184366070

Fig. 10.Compared between exact solution and HPM



$\overline{u}_1(x,\alpha) =$interval$[1 \quad \frac{1}{2-\alpha} \quad \alpha, \frac{2}{\alpha} \quad 2]$

$$\int_{1-\frac{1}{2-\alpha}}^{-0.25+\sqrt{1-\alpha}} k(x,t,\overline{k_1}(\overline{F}(t,\overline{u_0}(t,\alpha)) + \int_{-0.25+\sqrt{1-\alpha}}^{-0.5+\sqrt{1-\alpha}} k(x,t,k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u}_0(t,\alpha)\right) + \int_{-0.5+\sqrt{1-\alpha}}^{-0.75+\sqrt{1-\alpha}} k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u}_0(t,\alpha)\right) dt$$
$$+\int_{-0.75+\sqrt{1-\alpha}}^{\frac{2}{\alpha}-2} k(x,t,\underline{k_1}(t,\overline{F}(t,\overline{u_0}(t,\alpha))dt$$

$$x(2-r) - \frac{1}{5}x(2-r)^2\left(\left(0.25+\sqrt{1-r}\right)^5 - (1-r)^{5/2}\right) - \frac{1}{6}xr^2\left(\left(0.5+\sqrt{1-r}\right)^6 - \left(0.25+\sqrt{1-r}\right)^6\right) - \frac{1}{5}xr^2\left(\left(0.75+\sqrt{1-r}\right)^5 - \left(0.5+\sqrt{1-r}\right)^5\right) - \frac{1}{6}xr^2\left(\left(x-\sqrt{1-r}\right)^6 - \left(0.75+\sqrt{1-r}\right)^6\right)$$

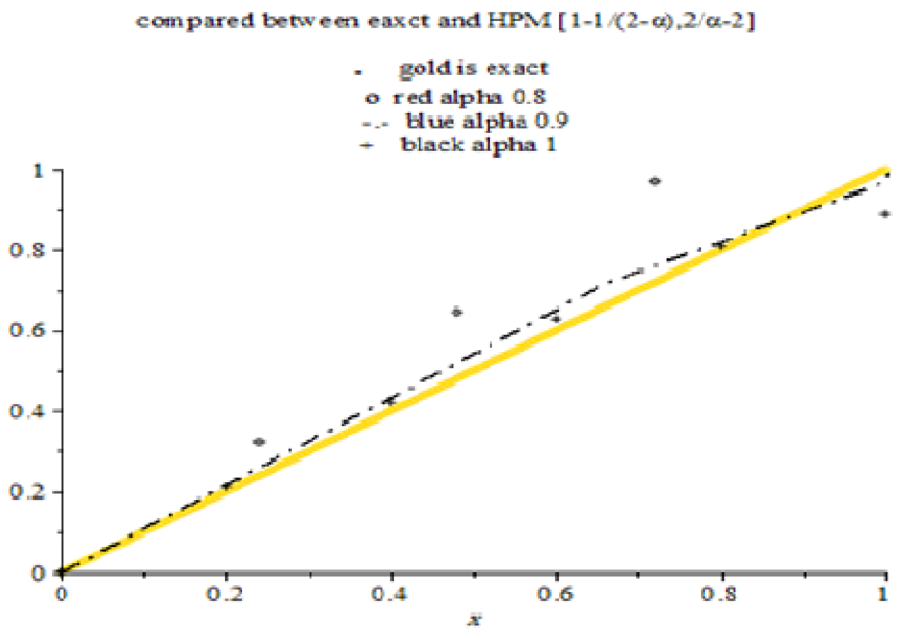
$\overline{u}_0(x,\alpha) = \overline{f}(x,\alpha) =$

$$\overline{u}(x,\alpha) = \overline{u}_0(x,\alpha) + p^1\overline{u}_1(x,\alpha) + \quad \dots$$

Table 11. For upper bound interval$(1 \quad \frac{1}{2-\alpha}, \frac{2}{\alpha} \quad 2)$

X	Exact $\alpha = 0.8$	HPM $\alpha = 0.8$	Exact $\alpha = 0.9$	HPM $\alpha = 0.9$	Exact $\alpha = 1$	HPM $\alpha = 1$
0	0.00000	0.0000000000	0.000000	0.0000000000	0.00000	0.0000000000
0.2	0.24000	0.2355658399	0.220000	0.2357630832	0.20000	0.2112887566
0.4	0.48000	0.5311535923	0.440000	0.4715263373	0.40000	0.4223086476
0.6	0.72000	0.7467285584	0.660000	0.6572263642	0.60000	0.6292663085
0.8	0.96000	1.2619370470	0.880000	0.9409846385	0.80000	0.8102068424
1	1.20000	1.5710361820	1.100000	1.1582047280	1.00000	0.8897729073

Fig.11. Compared between exact and HPM



$\overline{u}_1(x,\alpha)$ =in interval [1 $\frac{1}{2-\alpha}$ $\alpha,\frac{3}{\alpha}$ 3]

$\int_{1-1/2}^{-0.25+\sqrt{1-\alpha}} k(x,t,\overline{k_1}(\overline{F}(t,\overline{u_0}(t,\alpha)) + \int_{-0.25+\sqrt{1-\alpha}}^{-0.5+\sqrt{1-\alpha}} k(x,t,k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u}_0(t,\alpha)\right) + \int_{-0.5+\sqrt{1-\alpha}}^{-0.75+\sqrt{1-\alpha}} k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u}_0(t,\alpha)\right) dt$
$+\int_{-0.75+\sqrt{1-\alpha}}^{\frac{3}{\alpha}-3} k(x,t,\underline{k_1}(t,\overline{F}(t,\overline{u_0}(t,\alpha))dt$

$$\overline{u}_0(x,\alpha) = \overline{f}(x,\alpha) = \begin{aligned} & x(2-r) - \frac{1}{5}x(2-r)^2\left(\left(0.25+\sqrt{1-r}\right)^5 - (1-r)^{5/2}\right) - \frac{1}{6}xr^2\left(\left(0.5+\sqrt{1-r}\right)^6\right. \\ & \left.-\left(0.25+\sqrt{1-r}\right)^6\right) - \frac{1}{5}xr^2\left(\left(0.75+\sqrt{1-r}\right)^5 - \left(0.5+\sqrt{1-r}\right)^5\right) \\ & -\frac{1}{6}xr^2\left(\left(x-\sqrt{1-r}\right)^6 - \left(0.75+\sqrt{1-r}\right)^6\right) \end{aligned}$$

$$\overline{u}(x,\alpha) = \overline{u}_0(x,\alpha) + p^1\overline{u}_1(x,\alpha) + \quad \ldots..$$

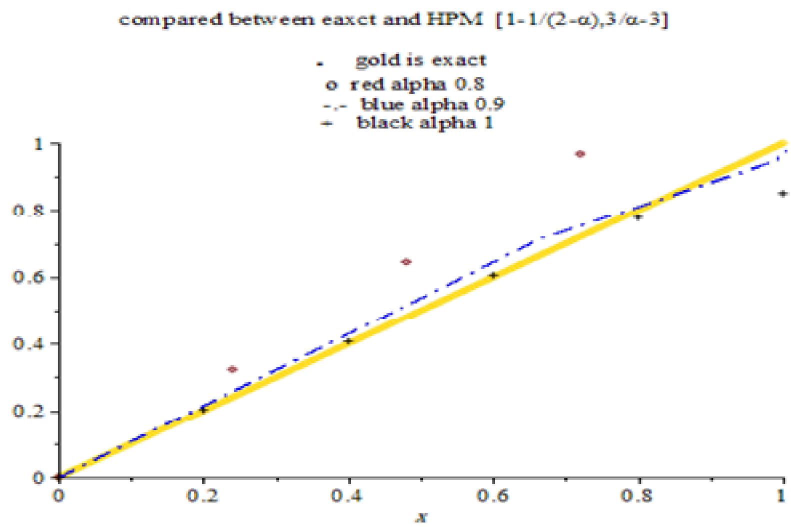
Table 12. For upper bound interval (1 $\frac{1}{2-\alpha},\frac{3}{\alpha}$ 3)

X	Exact $\alpha = 0.8$	HPM $\alpha = 0.8$	Exact $\alpha = 0.9$	HPM $\alpha = 0.9$	Exact $\alpha = 1$	HPM $\alpha = 1$
0	0.00000	0.0000000000	0.000000	0.0000000000	0.00000	0.0000000000
0.2	0.24000	0.3225929741	0.220000	0.2369682022	0.20000	0.2034439477
0.4	0.48000	0.6452078607	0.440000	0.4739365751	0.40000	0.4066196080
0.6	0.72000	0.9678099610	0.660000	0.7108417209	0.60000	0.6056734042
0.8	0.96000	1.2900455840	0.880000	0.9458051141	0.80000	0.7788359742
1	1.20000	1.6061718530	1.100000	1.1642303230	1.00000	0.8505786148

$\overline{u}_1(x,\alpha)$ =in interval [2 $\frac{2}{2-\alpha}$ $\alpha,\frac{1}{\alpha}$]

$\int_{2-\frac{2}{2-\alpha}}^{-0.25+\sqrt{1-\alpha}} k(x,t,\overline{k_1}(\overline{F}(t,\overline{u_0}(t,\alpha)) + \int_{-0.25+\sqrt{1-\alpha}}^{-0.5+\sqrt{1-\alpha}} k(x,t,k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u}_0(t,\alpha)\right) + \int_{-0.5+\sqrt{1-\alpha}}^{-0.75+\sqrt{1-\alpha}} k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u}_0(t,\alpha)\right) dt$
$+\int_{-0.75+\sqrt{1-\alpha}}^{\frac{1}{\alpha}} k(x,t,\underline{k_1}(t,\overline{F}(t,\overline{u_0}(t,\alpha))dt$

Fig.12. Compared between exact and HPM

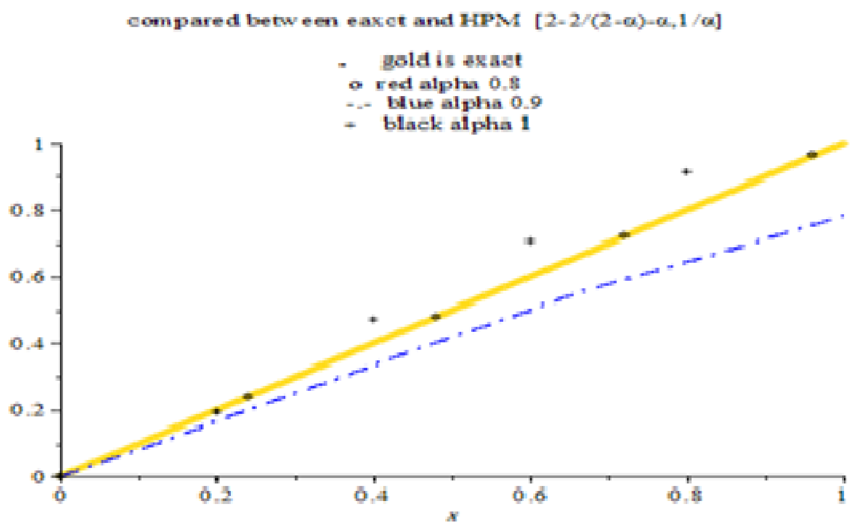


$$\overline{u}_0(x,\alpha)=\overline{f}(x,\alpha)= x(2-r)-\frac{1}{5}x(2-r)^2\left((0.25+\sqrt{1-r})^5-(1-r)^{5/2}\right)-\frac{1}{6}xr^2\left((0.5+\sqrt{1-r})^6-(0.25+\sqrt{1-r})^6\right)-\frac{1}{5}xr^2\left((0.75+\sqrt{1-r})^5-(0.5+\sqrt{1-r})^5\right)-\frac{1}{6}xr^2\left((x-\sqrt{1-r})^6-(0.75+\sqrt{1-r})^6\right)$$

Table 13. For upper bound interval$\left(2-\frac{2}{2-\alpha},\frac{1}{\alpha}\right)$

X	Exact $\alpha=0.8$	HPM $\alpha=0.8$	Exact $\alpha=0.9$	HPM $\alpha=0.9$	Exact $\alpha=1$	HPM $\alpha=1$
0	0.00000	0.0000000000	0.000000	0.0000000000	0.00000	0.0000000000
0.2	0.24000	0.2412112572	0.220000	0.1832768754	0.20000	0.1941272200
0.4	0.48000	0.4824450397	0.440000	0.3665539218	0.40000	0.4737623581
0.6	0.72000	0.7236657272	0.660000	0.5497677390	0.60000	0.7063882002
0.8	0.96000	0.9645197405	0.880000	0.7710397709	0.80000	0.9131223690
1	1.20000	1.1992624280	1.100000	0.8957734341	1.00000	1.0184366070

Fig. 13. Compared between exact and HPM



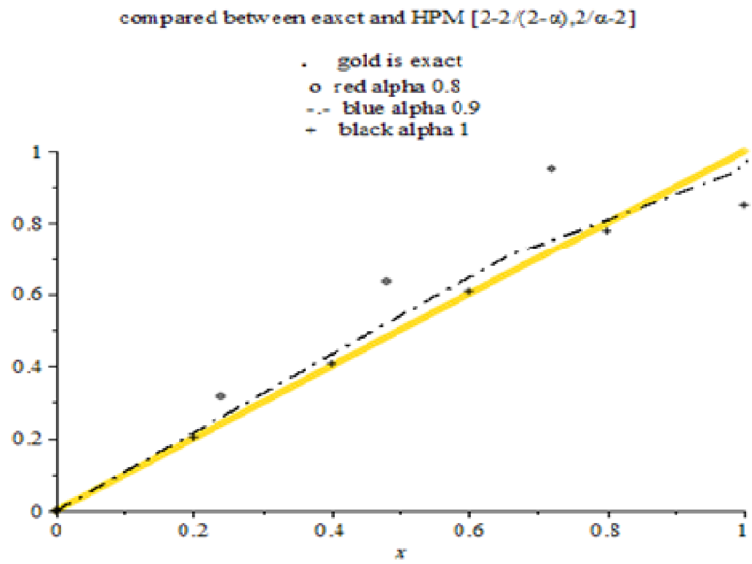
$$\int_{2-\frac{2}{2-\alpha}}^{-0.25+\sqrt{1-\alpha}} k(x,t,\overline{k_1}(\overline{F}(t,\overline{u_0}(t,\alpha))+\int_{-0.25+\sqrt{1-\alpha}}^{-0.5+\sqrt{1-\alpha}} k(x,t,k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u}_0(t,\alpha)\right)+\int_{-0.5+\sqrt{1-\alpha}}^{-0.75+\sqrt{1-\alpha}} k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u}_0(t,\alpha)\right)dt$$
$$+\int_{-0.75+\sqrt{1-\alpha}}^{\frac{2}{\alpha}-2} k(x,t,\underline{k_1}(t,\overline{F}\left(t,\overline{u_0}(t,\alpha)\right)dt$$

$$\overline{u}_0(x,\alpha) = \overline{f}(x,\alpha) = x(2-r) - \frac{1}{5}x(2-r)^2\left((0.25+\sqrt{1-r})^5 - (1-r)^{5/2}\right) - \frac{1}{6}xr^2\left((0.5+\sqrt{1-r})^6 - (0.25+\sqrt{1-r})^6\right) - \frac{1}{5}xr^2\left((0.75+\sqrt{1-r})^5 - (0.5+\sqrt{1-r})^5\right) - \frac{1}{6}xr^2\left((x-\sqrt{1-r})^6 - (0.75+\sqrt{1-r})^6\right)$$

Table 14. For upper bound interval$\left(2 \quad \frac{2}{2-\alpha}, \frac{2}{\alpha} \quad 2\right)$

6	Exact $\alpha = 0.8$	HPM $\alpha = 0.8$	Exact $\alpha = 0.9$	HPM $\alpha = 0.9$	Exact $\alpha = 1$	HPM $\alpha = 1$
0	0.00000	00.00000000	0.000000	0.0000000000	0.00000	0.0000000000
0.2	0.24000	0.3173362893	0.220000	0.2368530716	0.20000	0.2034439477
0.4	0.48000	0.6346944741	0.440000	0.4737663141	0.40000	0.4066191608
0.6	0.72000	0.9520398289	0.660000	0.7104963275	0.60000	0.6056734042
0.8	0.96000	1.269018609	0.880000	0.9453445555	0.80000	0.7788359742
1	1.20000	1.579880150	1.100000	1.1636544150	1.00000	0.8505786148

Fig.14. Compared between exact and HPM



$\overline{u}_1(x,\alpha)$ =in interval $\left[2 \quad \frac{2}{2-\alpha} \quad \alpha, \frac{3}{\alpha} \quad 3\right]$

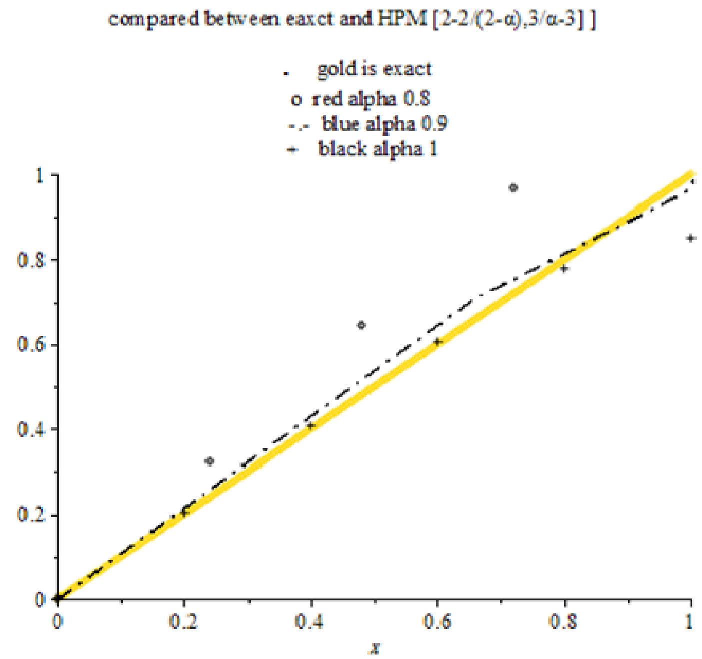
$$\int_{2-\frac{2}{2-\alpha}}^{-0.25+\sqrt{1-\alpha}} k(x,t,\overline{k_1}(\overline{F}(t,\overline{u_0}(t,\alpha)) + \int_{-0.25+\sqrt{1-\alpha}}^{-0.5+\sqrt{1-\alpha}} k(x,t,k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u}_0(t,\alpha)\right) + \int_{-0.5+\sqrt{1-\alpha}}^{-0.75+\sqrt{1-\alpha}} k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u}_0(t,\alpha)\right)dt$$
$$+\int_{-0.75+\sqrt{1-\alpha}}^{\frac{3}{\alpha}-3} k(x,t,\underline{k_1}(t,\overline{F}\left(t,\overline{u_0}(t,\alpha)\right)dt$$

$$\overline{u}_0(x,\alpha) = \overline{f}(x,\alpha) = x(2-r) - \frac{1}{5}x(2-r)^2\left((0.25+\sqrt{1-r})^5 - (1-r)^{5/2}\right) - \frac{1}{6}xr^2\left((0.5+\sqrt{1-r})^6 - (0.25+\sqrt{1-r})^6\right) - \frac{1}{5}xr^2\left((0.75+\sqrt{1-r})^5 - (0.5+\sqrt{1-r})^5\right) - \frac{1}{6}xr^2\left((x-\sqrt{1-r})^6 - (0.75+\sqrt{1-r})^6\right)$$

Table 15. For upper bound interval$\left(2 \quad \frac{2}{2-\alpha}, \frac{3}{\alpha} \quad 3\right)$

X	Exact $\alpha = 0.8$	HPM $\alpha = 0.8$	Exact $\alpha = 0.9$	HPM $\alpha = 0.9$	Exact $\alpha = 1$	HPM $\alpha = 1$
0	0.00000	0.0000000000	0.000000	0.0000000000	0.00000	0.0000000000
0.2	0.24000	0.3226218541	0.220000	0.2369691650	0.20000	0.2034439477
0.4	0.48000	0.6452656038	0.440000	0.4739385007	0.40000	0.4066191608
0.6	0.72000	0.9678965734	0.660000	0.7108446075	0.60000	0.6056734042
0.8	0.96000	1.2901608690	0.880000	0.9458089288	0.80000	0.7788359742
1	1.20000	1.6063138390	1.100000	1.1642348820	1.00000	0.8505786148

Fig.15. Compared between exact and HPM



$\overline{u}_1(x,\alpha)$ =in interval [3 $\frac{3}{2-\alpha}$ $\alpha,\frac{1}{\alpha}$]

$$\int_{3-\frac{3}{2-\alpha}}^{-0.25+\sqrt{1-\alpha}} k(x,t,\overline{k_1}(\overline{F}(t,\overline{u_0}(t,\alpha)) + \int_{-0.25+\sqrt{1-\alpha}}^{-0.5+\sqrt{1-\alpha}} k(x,t,k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u}_0(t,\alpha)\right) + \int_{-0.5+\sqrt{1-\alpha}}^{-0.75+\sqrt{1-\alpha}} k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u}_0(t,\alpha)\right)dt$$
$$+\int_{-0.75+\sqrt{1-\alpha}}^{\frac{1}{\alpha}} k(x,t,\underline{k_1}(t,\overline{F}(t,\overline{u_0}(t,\alpha))dt$$

$$\overline{u}_0(x,\alpha) = \overline{f}(x,\alpha) = x(2-r) - \frac{1}{5}x(2-r)^2\left(\left(0.25+\sqrt{1-r}\right)^5 - (1-r)^{5/2}\right) - \frac{1}{6}xr^2\left(\left(0.5+\sqrt{1-r}\right)^6 - \left(0.25+\sqrt{1-r}\right)^6\right) - \frac{1}{5}xr^2\left(\left(0.75+\sqrt{1-r}\right)^5 - \left(0.5+\sqrt{1-r}\right)^5\right) - \frac{1}{6}xr^2\left(\left(x-\sqrt{1-r}\right)^6 - \left(0.75+\sqrt{1-r}\right)^6\right)$$

Table 16. For upper bound interval (3 $\frac{3}{2-\alpha},\frac{1}{\alpha}$)

X	Exact $\alpha = 0.8$	HPM $\alpha = 0.8$	Exact $\alpha = 0.9$	HPM $\alpha = 0.9$	Exact $\alpha = 1$	HPM $\alpha = 1$
0	0.00000	0.0000000000	0.000000	0.0000000000	0.00000	0.0000000000
0.2	0.24000	0.2415164175	0.220000	0.1833870386	0.20000	0.2370155464
0.4	0.48000	0.4805455060	0.440000	0.3665742478	0.40000	0.4894184899
0.6	0.72000	0.7245799693	0.660000	0.5947982028	0.60000	0.7298723978
0.8	0.96000	0.9657366339	0.880000	0.7831080036	0.80000	0.9444346324
1	1.20000	1.2007611690	1.100000	0.8958215553	1.00000	1.0575769370

$\overline{u}_1(x,\alpha)$ =in interval [3 $\frac{3}{2-\alpha}$ $\alpha,\frac{2}{\alpha}$ 2]

$$\int_{3-\frac{3}{2-\alpha}}^{-0.25+\sqrt{1-\alpha}} k(x,t,\overline{k_1}(\overline{F}(t,\overline{u_0}(t,\alpha)) + \int_{-0.25+\sqrt{1-\alpha}}^{-0.5+\sqrt{1-\alpha}} k(x,t,k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u}_0(t,\alpha)\right) + \int_{-0.5+\sqrt{1-\alpha}}^{-0.75+\sqrt{1-\alpha}} k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u}_0(t,\alpha)\right)dt$$
$$+\int_{-0.75+\sqrt{1-\alpha}}^{\frac{2}{\alpha}-2} k(x,t,\underline{k_1}(t,\overline{F}(t,\overline{u_0}(t,\alpha))dt$$

$$\overline{u}_0(x,\alpha) = \overline{f}(x,\alpha) = x(2-r) - \frac{1}{5}x(2-r)^2\left(\left(0.25+\sqrt{1-r}\right)^5 - (1-r)^{5/2}\right) - \frac{1}{6}xr^2\left(\left(0.5+\sqrt{1-r}\right)^6 - \left(0.25+\sqrt{1-r}\right)^6\right) - \frac{1}{5}xr^2\left(\left(0.75+\sqrt{1-r}\right)^5 - \left(0.5+\sqrt{1-r}\right)^5\right) - \frac{1}{6}xr^2\left(\left(x-\sqrt{1-r}\right)^6 - \left(0.75+\sqrt{1-r}\right)^6\right)$$

Fig.16. Compared between exact and HPM

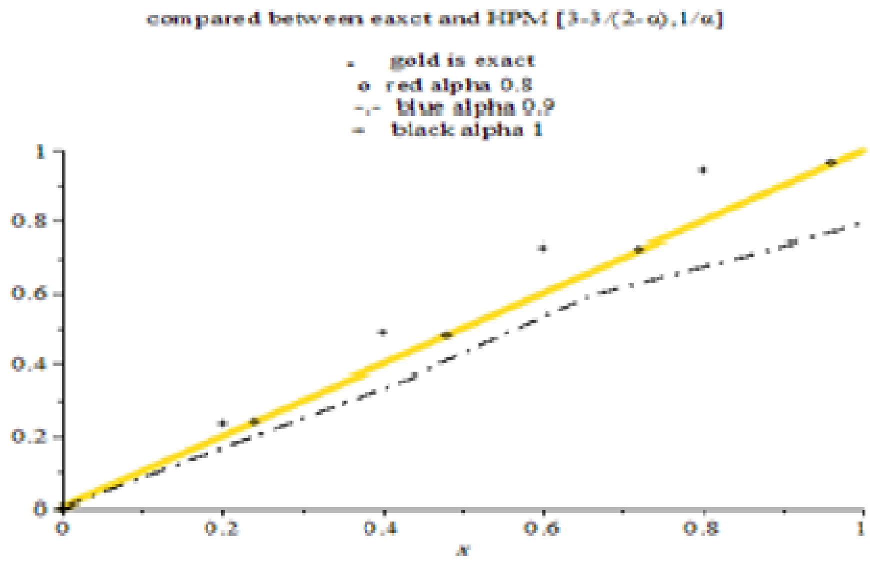
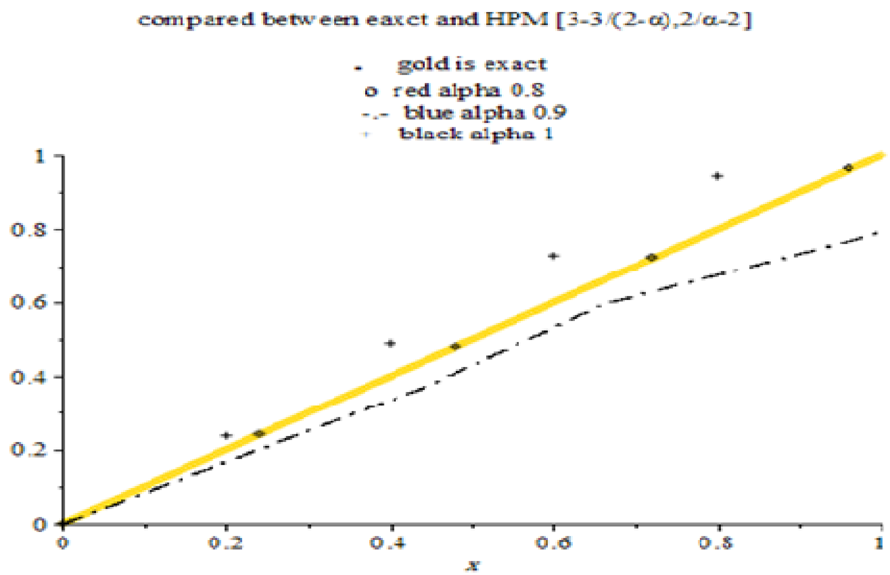


Table 17. For upper bound interval$(3\ \ \frac{3}{2-\alpha},\frac{2}{\alpha}\ \ 2)$

X	Exact $\alpha = 0.8$	HPM $\alpha = 0.8$	Exact $\alpha = 0.9$	HPM $\alpha = 0.9$	Exact $\alpha = 1$	HPM $\alpha = 1$
0	0.00000	0.0000000000	0.000000	0.0000000000	0.00000	0.0000000000
0.2	0.24000	0.3158995633	0.220000	0.2357742090	0.20000	0.1829843206
0.4	0.48000	0.6318208463	0.440000	0.4715485889	0.40000	0.3659990600
0.6	0.72000	0.9477294128	0.660000	0.7077597195	0.60000	0.5442945229
0.8	0.96000	1.2632692250	0.880000	0.9410287187	0.80000	0.6969974660
1	1.20000	1.5726769090	1.100000	1.1582574080	1.00000	0.8897189433

Fig.17. Compared between exact and HPM



$\overline{u}_1(x,\alpha)$ =in interval $[3\ \ \frac{3}{2-\alpha}\ \ \ \alpha,\frac{3}{\alpha}\ \ \ 3]$

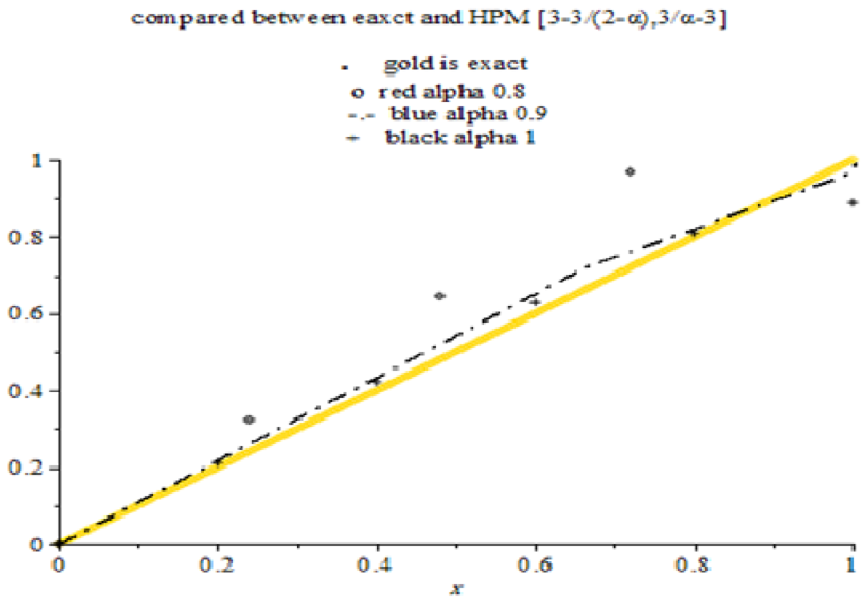
$$\int_{3-\frac{3}{2-\alpha}}^{-0.25+\sqrt{1-\alpha}} k(x,t,\overline{k_1}(\overline{F}(t,\overline{u_0}(t,\alpha)) + \int_{-0.25+\sqrt{1-\alpha}}^{-0.5+\sqrt{1-\alpha}} k(x,t,k(x,t,\underline{k_1}(t,\underline{F}\left(t,\underline{u}_0(t,\alpha)\right) + \int_{-0.5+\sqrt{1-\alpha}}^{-0.75+\sqrt{1-\alpha}} k(x,t,\overline{k_1}(t,\underline{F}\left(t,\underline{u}_0(t,\alpha)\right)dt$$
$$+\int_{-0.75+\sqrt{1-\alpha}}^{\frac{3}{\alpha}-3} k(x,t,\underline{k_1}(t,\overline{F}\left(t,\overline{u_0}(t,\alpha)\right)dt$$

$$\overline{u}_0(x,\alpha)=\overline{f}(x,\alpha)= x(2-r)-\frac{1}{5}x(2-r)^2\left((0.25+\sqrt{1-r})^5-(1-r)^{5/2}\right)-\frac{1}{6}xr^2\left((0.5+\sqrt{1-r})^6-(0.25+\sqrt{1-r})^6\right)-\frac{1}{5}xr^2\left((0.75+\sqrt{1-r})^5-(0.5+\sqrt{1-r})^5\right)-\frac{1}{6}xr^2\left((x-\sqrt{1-r})^6-(0.75+\sqrt{1-r})^6\right)$$

Table 18. For upper bound interval$(3 \;\frac{3}{2-\alpha},\frac{3}{\alpha}\; 3)$

X	Exact $\alpha = 0.8$	HPM $\alpha = 0.8$	Exact $\alpha = 0.9$	HPM $\alpha = 0.9$	Exact $\alpha = 1$	HPM $\alpha = 1$
0	0.00000	0.0000000000	0.000000	0.0000000000	0.00000	0.0000000000
0.2	0.24000	0.3229905027	0.220000	0.2375496642	0.20000	0.2112720136
0.4	0.48000	0.6460027211	0.440000	0.4750994989	0.40000	0.4222252925
0.6	0.72000	0.9690022249	0.660000	0.7175860846	0.60000	0.6291576019
0.8	0.96000	1.2916329750	0.880000	0.9481305388	0.80000	0.8101482378
1	1.20000	1.5081315960	1.100000	1.1671346830	1.00000	0.8897189433

Fig.18. Compared between exact and HPM



6- Conclusion

In this paper, the homotopy perturbation method has been successfully applied to find the solution of nonlinear fuzzy volterra integral classes equation over fuzzy interval. It is apparently seen that HPM is a very powerful and efficient technique in finding analytical solutions for complex classes of nonlinear problems. It is worth pointing out that this method presents a convergence for the solutions. The computations associated with the examples in this letter were performed using Maple 18.

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