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RESEARCH ARTICLE

SCHUR COMPLEMENTS PIVOTAL TRANSFORMATION ON CON-s-k-EP MATRICES

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ABSTRACT

Necessary and sufficient conditions are determined for a schur complement Pivotal Transformation in a con-s-k-EP matrix to be con-s-k-EP. Further it is shown that in a con-s-k-EP_r Matrix, every secondary sub matrix of rank 'r' is con-s-k-EP_r. Also discussed the question of expressing a matrix of rank r as a product of con-s-k-EP_r matrix. A necessary and sufficient condition for products of con-s-k-EPr Partitioned matrices to be con-s-k-EPr is given.
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INTRODUCTION

Throughout we shall deal with C_{nxn} the space of nxn complex matrices. Let C_n be the space of complex n-tuples. For $A \in C_{nxn}$, let A^T A^* and $A^\dagger$ denotes the transpose, conjugate transpose and Moore Penrose inverse of A respectively. A matrix A is called con-s-k-EP_r if $\rho(A) = r$ and $N(A) = N(A^T VK)$ or $R(A) = R(KVA^T)$ where $\rho(A)$ denotes the rank of A, N(A) and R(A) denotes the null space and range space of A respectively. Throughout let 'k' be the fixed product of disjoint transposition in S_n=1, 2, n and K be the associated permutation matrix. Let us define the function $k(x) = (x_{k(1)}, x_{k(2)}, ..., x_{k(n)})$. A matrix $A=(a_{ij}) \in C_{nxn}$ is s-k symmetric if $a_{ij} = a_{n-k(j)+1,n-k(i)+1}$ for i,j=1,2,….n. A matrix $A \in C_{nxn}$ is said to be con-s-k-EP if it satisfies the condition $Ax=0 \Leftrightarrow A^S k (x) = 0$ or equivalently $N(A) = N(A^T VK)$. In addition to that A is con-s-k-EP $\Leftrightarrow$ KVA is con-EP or AVK is con-EP and A is con-s-k-EP $\Leftrightarrow$ A^T is con-s-k-EP. Moreover A is said to be con-s-k-EP_r if A is con-s-k-EP and of rank r. For further properties of con-s-k-EP matrices one may refer [3]. In this paper we give necessary and sufficient conditions for a schur complement pivotal transformation in a con-s-k-EP matrix to be con-s-k-EP. Further it is shown that in a con-s-k- EP_r matrix, every secondary sub matrix of rank r is con-s-k-EP_r. Also discussed the question of expressing a matrix of rank r as a product of con-s-k-EP_r matrices. Necessary and sufficient conditions for products of con-s-k-EP_r partitioned matrices to be con-s-k-EP_r are given. In this sequel, we need the following theorems.

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Theorem 1.1[2]. For $A, B \in C_{nxn}$, the following hold:

(i) $\rho(AA^*) = \rho(A^*A) = \rho(A) = \rho(A^T)$
$= \rho(A^*) = \rho(\overline{A}) = \rho(A^\dagger)$

(ii) $\rho(AB) = \rho(B) - \dim N((A) \cap N(B^*)^\perp)$

Theorem 1.2[1]. Let $A, B \in C_{nxn}$, and $U \in C_{nxn}$ be any non singular matrix. Then,

(i) $R(A) = R(B) \Leftrightarrow R(UAU^*) = R(UBU^*)$

(ii) $N(A) = N(B) \Leftrightarrow N(UAU^*) = N(UBU^*)$

Theorem 1.3 [9]. Let $A, B \in C_{nxn}$, Then

(i) $N(A) \subseteq N(B) \Leftrightarrow R(B^*) \subseteq R(A^*)$
$\Leftrightarrow B = BA^- A$ for all $A^- \in A\{1\}$

(ii) $N(A^*) \subseteq N(B^*) \Leftrightarrow R(B) \subseteq R(A)$
$\Leftrightarrow B = AA^- B$ for every $A^- \in A\{1\}$

Definition 1.4[3]. Let M be an nxn matrix of the form $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. A schur complement of A in M is $(M/A) = D - CA^- B$.

Theorem 1.5 (Theorem 1 [4]). Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, Then

$$\rho(M) \geq \rho(A) + \rho(M/A)$$

With equality if and only if

$$N(M/A) \subseteq N\left(\left(I - AA^{\dagger}\right)B\right)$$

$$N(M/A)^{*} \subseteq N\left(\left(I - A^{\dagger}A\right)C^{*}\right) \text{ and}$$

$$\left(I - AA^{\dagger}\right)B(M/A)^{\dagger} \subset \left(I - A^{\dagger}A\right) = 0$$

In particular, we have the equality if M satisfies

$N(A) \subseteq N(C)$ and $N(A^{*}) \subseteq N(B^{*})$.

Theorem 1.6(Theorem 1 [3] and [8]). Let

$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, Then

$$M^{\dagger} = \begin{pmatrix} A^{\dagger} + A^{\dagger}B(M/A)^{\dagger}CA & -A^{\dagger}B(M/A)^{\dagger} \\ -(M/A)^{\dagger}CA^{\dagger} & (M/A)^{\dagger} \end{pmatrix}$$

$$\Leftrightarrow N(A) \subseteq N(C), N(A^{*}) \subseteq N(B^{*}),$$

$$N(M/A)^{*} \subseteq N(C^{*}) \text{ and } N(M/A) \subseteq N(B)$$

Also, $M^{\dagger} = \begin{pmatrix} (M/D)^{\dagger} & -A^{\dagger}B(M/A)^{\dagger} \\ -D^{\dagger}C(M/D)^{\dagger} & (M/A)^{\dagger} \end{pmatrix}$

$$\Leftrightarrow N(A) \subseteq N(C), N(A^{*}) \subseteq N(B^{*}),$$

$$N(M/D)^{*} \subseteq N(B^{*})\ N(M/D) \subseteq N(C).$$

When, $\rho(M) = \rho(A)$ then $M = \begin{pmatrix} A & B \\ C & CA^{-}B \end{pmatrix}$ and

$M = \begin{pmatrix} A^{*}PA^{*} & A^{*}PC^{*} \\ B^{*}PA^{*} & B^{*}PC^{*} \end{pmatrix}$ where

$$P = \left(AA^{*} + BB^{*}\right)^{-} A\left(A^{*}A + C^{*}C\right)^{-}$$

2. PIVOTAL TRANSFORMATION ON CON-S-K-EP MATRICES

In this section we have given necessary and sufficient conditions for a con-s-k-EP matrix to have its secondary sub matrices and their schur complement to be con-s-k-EP. This is a generalization of the result found in [7]. As an application it is shown that the property of a matrix being con-s-k-EP$_r$ is persevered under the secondary pivot transformation. It is well known that Theorem (1.2) [7], the class of con-EP matrices is invariant under secondary rearrangement. By a secondary rearrangement of a sequence matrix M, we mean a matrix $P^T MP$

Where P is a permutation matrix $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. By a secondary rearrangement of a square matrix M, we mean a matrix $P^T VMP$. Similarly the secondary k rearrangement of a square matrix M we mean a matrix $P^T KVMP$.

Let M be a matrix of the form $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$

(2.1)

and let S be a matrix of the form $S = \begin{pmatrix} (M/A) & (M/B) \\ (M/C) & (M/D) \end{pmatrix}$

(2.2)

$K = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$ and $V = \begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix}$. Now,

$$KVS = \begin{bmatrix} K_1 v(M/C) & K_1 v(M/D) \\ K_2 v(M/A) & K_2 v(M/B) \end{bmatrix}$$

Then,

$$P^T(KVS)P = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{bmatrix} K_1 v(M/C) & K_1 v(M/D) \\ K_2 v(M/A) & K_2 v(M/B) \end{bmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

$$= \begin{bmatrix} K_2 v(M/A) & K_2 v(M/B) \\ K_1 v(M/C) & K_1 v(M/D) \end{bmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

$$= \begin{bmatrix} K_2 v(M/B) & K_2 v(M/A) \\ K_1 v(M/D) & K_1 v(M/C) \end{bmatrix}$$

Let us consider a system of linear equation SZ = t where S is of the form [2.2] satisfy $N(M/C) \subseteq N(M/A)$ and $N(M/C)^T \subseteq N(M/D)^T$. If Z and t are partitioned conformably as $Z = \begin{bmatrix} x \\ y \end{bmatrix}$ and $t = \begin{bmatrix} u \\ w \end{bmatrix}$ then the system becomes $\begin{bmatrix} (M/A) & (M/B) \\ (M/C) & (M/D) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ w \end{bmatrix}$

$$\Rightarrow \quad (M/A)\,x + (M/B)\,y = u$$

$$(M/C)\,x + (M/D)\,y = w$$

Since S satisfies $N(M/C) \subseteq N(M/A)$ and $N(M/C)^T \subseteq N(M/D)^T$.

Using $(M/A) = (M/A)(M/C)^{-}(M/C)$ and $(M/D) = (M/C)^{-}(M/C)(M/D)$ (Theorem (1.3)) we can solve **x** and *w* as,

$$x = (M/C)^{\dagger}u - (M/C)^{\dagger}(M/D)\,y\,;$$

$$w = (M/A)(M/C)^{\dagger}u + (M/B) - (M/A)(M/C)^{\dagger}(M/D)\,y.$$

Thus a matrix S of the form (2.2), that satisfies $N(M/C) \subseteq N(M/A)$ and $N(M/C)^T \subseteq N(M/D)^T$ can be transformed into the matrix

$$\tilde{S} = \begin{bmatrix} (M/C)^{\dagger} & -(M/C)^{\dagger}(M/D) \\ (M/A)(M/C)^{\dagger} & [S/(M/D)] \end{bmatrix} \qquad (2.3)$$

$\tilde{S}$ is called a secondary pivot transform of S. The operation that transforms $S \to \tilde{S}$ is called secondary pivot.

Lemma 2.1. Let S be a matrix of the form (2.2) with $N(M/C) \subseteq N(M/A)$ and $N(M/B) \subseteq N(M/D)$. Then the following are equivalent.

(i) S is con-s-k-EP with $k = k_1 k_2$ Where

$$K = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}, \quad V = \begin{bmatrix} 0 & \nu \\ \nu & 0 \end{bmatrix},$$

$N[S/(M/C)] \subseteq N(M/D)$ and

$N[S/(M/B)] \subseteq N(M/A)$

(ii) (M/C) and $[S/(M/D)]$ are con-s-k_1-EP, (M/B) and $[S/(M/C)]$ are con-s-k_2-EP.

Further

$$N(M/C) = N[S/(M/B)] \subseteq N((M/D)^T \nu K_1)$$

and

$$N(M/B) = N[S/(M/C)] \subseteq N((M/A)^T \nu K_2).$$

Proof. Since S is con-s-k-EP with $k = k_1 k_2$ where

$$K = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}, V = \begin{bmatrix} 0 & \nu \\ \nu & 0 \end{bmatrix},$$

$N(M/C) \subseteq N(M/A)$ and $N[S/(M/C)] \subseteq N(M/D)$ (by Theorem 2.5 [5]) (M/C) is con-s-k_1-EP $[S/(M/C)]$ is con-s-k_2-EP.

$N(M/C) = N\left((M/D)^T \nu K_1\right)$ and

$N\left([S/(M/C)]^T \nu K_2\right) \subseteq N\left((M/A)^T \nu K_2\right)$.

Since (M/C) is con-s-k_1-EP,

$N\left((M/C)^T \nu K_2\right) = N(M/C)$ [by definition of con-s-k-EP matrix]. Therefore

$N\left((M/C)^T \nu K_2\right) \subseteq N\left((M/D)^T \nu K_2\right)$. Since S is con-s-k-EP, KVS is con-EP implies the secondary rearrangement

$$P^T KVSP = \begin{bmatrix} K_2\nu(M/B) & K_2\nu(M/A) \\ K_1\nu(M/D) & K_1\nu(M/C) \end{bmatrix}$$ is also con-EP.

Further $N\left(K_2\nu(M/B)\right) \subseteq N\left(K_1\nu(M/D)\right)$ and

$N\left(K_1\nu[S/(M/B)]\right) \subseteq N\left(K_2\nu(M/A)\right)$ hold. Hence by (Theorem (2.5) [6]) $K_2\nu(M/B)$ is con-EP.

$K_1\nu[S/(M/B)]$ is con-EP,

$N\left(K_2\nu(M/B)\right)^T \subseteq N\left(K_2\nu(M/A)\right)^T$ and

$N\left(K_1\nu\left[S/\left(M/B\right)\right]\right)^T \subseteq N\left(K_2\nu(M/D)\right)^T$. Thus we have (M/B) is con-s-k_2-EP, $\left[S/\left(M/B\right)\right]$ is con-s-k_1-EP

$N\left((M/B)^T \nu K_2\right) \subseteq N\left((M/A)^T \nu K_2\right)$ and

$N\left[S/\left(M/B\right)\right] \subseteq N\left((M/D)^T \nu K_1\right)$.

Since (M/B) is con-s-k_2-EP. By definition

$N\left((M/B)^T \nu K_2\right) = N(M/B)$.

Thus $N(M/B) \subseteq N\left((M/A)^T \nu K_2\right)$. Since the relations

$N(M/C) \subseteq N(M/A)$,

$N\left((M/C)^T \nu K_1\right) \subseteq N\left((M/D)^T \nu K_1\right)$,

$N[S/(M/C)] \subseteq N(M/D)$ and

$N\left([S/(M/B)]^T \nu K_2\right) \subseteq N\left((M/A)^T \nu K_2\right)$ holds for K1v (M/A) according to the assumptions (by Theorem (1.6)) $(KVS)^\dagger$ is given by the form

$$(KVS)^\dagger = \begin{bmatrix} (K_1\nu(M/C))^\dagger + (K_1\nu(M/C))^\dagger (K_1\nu(M/D))\left(K_2\nu\left[S/(M/C)\right]^\dagger\right)(K_2\nu(M/A))(K_1\nu(M/C))^\dagger & -(K_1\nu(M/C))^\dagger (K_1\nu(M/D))\left(K_2\nu\left[S/(M/C)\right]^\dagger\right) \\ -\left(K_2\nu[S/(M/C)]^\dagger\right)(K_2\nu(M/A))(K_2\nu(M/C))^\dagger & (K_2\nu[S/(M/C)])^\dagger \end{bmatrix}$$

By using

$K_2\nu(M/A) = K_2\nu[S/(M/C)]\left(K_2\nu[S/(M/C)]\right)^\dagger (K_2\nu(M/A))$ and

$K_1\nu(M/D) = K_1\nu(M/C)\left(K_1\nu(M/C)\right)^\dagger \left(K_1\nu(M/D)\right)$,

$(KVS)(KVS)^\dagger$ reduces to the form

$$(KVS)(KVS)^\dagger = \begin{pmatrix} (K_1\nu(M/C))\left(K_1\nu(M/C)^\dagger\right) & 0 \\ 0 & (K_2\nu[S/(M/C)])\left(K_2\nu[S/(M/C)]^\dagger\right) \end{pmatrix}$$

(2.5)

Since the relation $N(M/B) \subseteq N(M/D)$,

$N\left((M/B)^T \nu K_2\right) \subseteq N\left(A^T \nu K_2\right)$,

$N[S/(M/B)] \subseteq N(M/A)$ and

$N\left([S/(M/B)]^T \nu K_1\right) \subseteq N\left((M/D)^T \nu K_1\right)$ holds for $K_1\nu(M/B)$ according to the assumptions (by Theorem (1.6)) $(KVS)^\dagger$ is also given by the formula,

$$(KVS)^\dagger = \begin{bmatrix} K_1\nu[S/(M/C)]^\dagger & -(K_1\nu(M/C))\,(K_1\nu(M/D))\left(K_2\nu[S/(M/C)]\right)^\dagger \\ -\left(K_2\nu(M/B)\right)(K_2\nu(M/A))\left(K_1\nu[S/(M/C)]\right)^\dagger & \left(K_2\nu[S/(M/C)]\right)^\dagger \end{bmatrix}$$

(2.6)

Further using

$\left(K_1\nu(M/D)\right) = \left(K_1\nu(M/C)\right)\left(K_1\nu(M/C)\right)^\dagger \left(K_1\nu(M/D)\right)$

that is, $(M/D) = (M/C)(M/C)^\dagger (M/D)$ and

$K_2\nu(M/A) = \left(K_2\nu(M/B)\right)\left(K_2\nu(M/B)\right)^\dagger \left(K_2\nu(M/A)\right)$

that is

$(M/A) = (M/B)(M/B)^\dagger (M/A)$ in [2.6]

$(KVS)(KVS)^\dagger$ reduces to the form,

$$\begin{pmatrix} (K_1\nu[S/(M/B)])\,(K_1\nu[S/(M/B)])^\dagger & 0 \end{pmatrix}$$

(2.7)

Comparing (2.5) and (2.7) we get,

$\left(K_1\nu(M/C)\right)\left(K_1\nu(M/C)\right)^\dagger = \left(K_1\nu\left[S/(M/B)\right]\right)\left(K_1\nu\left[S/(M/B)\right]\right)^\dagger$

$$K_1\nu(M/C)(M/C)^\dagger \nu K_1 = K_1\nu[S/(M/B)][S/(M/B)]^\dagger \nu K_1$$

$(M/C)(M/C)^\dagger = [S/(M/B)][S/(M/B)]^\dagger$ Since (M/C) and [S/ (M/B)] are con-s-k_1-EP.

$$(K_1\nu(M/C))^\dagger(K_1\nu(M/C)) = (K_1\nu[S/(M/B)])^\dagger(K_1\nu[S/(M/B)])$$

$$(M/C)^\dagger \nu K_1K_1\nu(M/C) = [S/(M/B)]^\dagger \nu K_1K_1\nu[S/(M/B)]$$

$$(M/C)^\dagger(M/C) = [S/(M/B)]^\dagger[S/(M/B)]$$

$$N(M/C)\subseteq N[S/(M/B)]$$

Similarly, by using the formula (2.5) and (2.7), we obtain the expressions for $(KVS)^\dagger(KVS)$. Comparing, these yields $(M/B)^\dagger(M/B) = [S/(M/C)]^\dagger[S/(M/C)]$ which implies $N(M/B) = N[S/(M/C)]$. Thus [ii] holds, (ii) $\Rightarrow$ (i)

$N[S/(M/C)]\subseteq N(M/D)$ follows directly from

$$N[S/(M/C)] = N(M/B)\subseteq N(M/D).$$

Similarly

$$N[S/(M/A)]\subseteq N(M/A)$$

follows from $N[S/(M/B)] = N(M/A)\subseteq N(M/A)$. Now (M/C) is con-s-$k_1$- EP and $[S/(M/C)]$ is con-s-k_2-EP satisfying the relation

$N(M/C)\subseteq N(M/A)$,

$N((M/C)^T\nu K_1)\subseteq N((M/D)^T\nu K_1)$,

$N[S/(M/C)]\subseteq N(M/D)$ &

$N([S/(M/C)]^T\nu K_2)\subseteq N((M/A)^T\nu K_2)$. Hence (by Theorem (2.4) [6]) S is con-s-k-EP, Thus [i] holds.

Theorem 2.8. Let S be a con-s-k-EP_r matrix of the form [2.2] with $k = k_1 k_2$ where $K = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}$ and $V = \begin{pmatrix} 0 & \nu \\ \nu & 0 \end{pmatrix}$, $N(M/C)\subseteq N(M/A)$, $N(M/B)\subseteq N(M/D)$, $N[S/(M/C)]\subseteq N(M/D)$ and $N[S/(M/B)]\subseteq N(M/A)$. Then the following holds.

(i) The secondary sub matrix (M/C) is con-s-k_1-EP and secondary sub matrix (M/B) is con-s-k-$_2$-EP.

(ii) The schur complement $[S/(M/C)]$ is con-s-k_2-EP and $[S/(M/B)]$ is con-s-k_1-EP.

(iii) Each secondary pivot transform of S is con-s-k_2-EP_r

Proof. (i) and (ii) are consequences of Lemma 2.5. By Lemma 2.5. KVS satisfies

$N(K_1\nu(M/C))\subseteq N(K_2\nu(M/A))$ and $N((M/C)^T\nu K_1)\subseteq N((M/D)^T\nu K_1)$ hence by

pivoting the block K_1VC, the secondary pivot transform $\tilde{S}$ of S is of the form,

$$□KVS = \begin{pmatrix} (K_1\nu(M/C))^\dagger & -(K_1\nu(M/C))^\dagger(K_1\nu(M/D)) \\ K_2\nu[S/(M/A)]\ (K_1\nu[S/(M/C)]^\dagger) & K_2\nu[S/(M/C)] \end{pmatrix}$$

$$□KVS = \begin{pmatrix} (M/C)^\dagger\nu K_1 & -(M/C)^\dagger(M/D) \\ K_2(M/A)(M/C)^\dagger K_1 & K_2\nu[S/(M/C)] \end{pmatrix}$$

In □KVS

$$N((M/C)^\dagger\nu K_1)\subseteq N(K_2\nu(M/A)(M/C)^\dagger\nu K_1) =$$

$$N((M/A)(M/C)^\dagger\nu K_1),$$

$$N((M/C)^\dagger\nu K_1)^T\subseteq N((M/C)^\dagger(M/D))^T$$

Further,

$$N(□KVS/(K_1\nu(M/C))^\dagger) = (K_2\nu[S/(M/C)]) + (K_2\nu(M/A)(M/C)^\dagger\nu K_2)$$

$$((M/C)^\dagger\nu K_1)((M/C)^\dagger(M/D))$$

$$= K_1\nu[S/(M/C)] + K_2\nu(M/A)(M/C)^\dagger(M/C)(M/C)^\dagger(M/D)$$

$$= K_2\nu[S/(M/C)] + K_2\nu(M/A)(M/C)^\dagger(M/D)$$

$$= K_2\nu([S/(M/C)] + (M/A)(M/C)^\dagger(M/D))$$

$$= K_2\nu(M/D)$$

$$\Rightarrow (□KVS/(K_1\nu(M/C))^\dagger) = K_2\nu[\tilde{S}/(M/C)^\dagger] = K_2\nu(M/D)$$

By the assumption

$N(K_2\nu[\tilde{S}/(M/C)^\dagger]) = N(K_2\nu(M/B))$ which implies $N[\tilde{S}/(M/C)^\dagger] = N(M/B)\subseteq N(M/D)$.

From Lemma 2.5. (M/C) is con-s-k_1-EP and (M/B) is con-s-k_2-EP, Therefore $(M/C)^\dagger$ is con-s-k_1-EP $[\tilde{S}/(M/C)^\dagger]$ is con-s-k_2-EP (By Theorem 2.11 [5]) and

Also $N(K_2\nu\tilde{S}/(M/C)^\dagger) = N(K_2\nu(M/B))^T$

$$N([\tilde{S}/(M/C)^\dagger]^T\nu K_2) = N((M/B)^T\nu K_2)\subseteq N((M/A)^T\nu K_2)$$

Now by applying Theorem (2.4), we are that $\tilde{S}$ is con-s-k-EP. Now

$$r=\rho(S)=\rho(M/C)+\rho[S/(M/C)]$$

(Theorem 1.5.)

$$=\rho(M/C)^{\dagger}+\rho(M/B)$$

(Theorem 1.1. & by Lemma 2.5.)

$$=\rho(M/C)^{\dagger}+\rho\left[\tilde{S}/(M/C)^{\dagger}\right]$$

$$=\rho(\tilde{S})$$

(Theorem 1.5.)

Thus $(\tilde{S})$ is con-s-k-EP$_r$. Similarly under the conditions given on S it can be transformed to its secondary Pivot transform by pivoting the block $K_1V(M/B)$ without changing the rand.

Remark 2.9. For k(i) = I, the identity transposition Theorem [2.8] reduces to the results for con-s-EP matrices. It KV=I then Theorem 2.8 reduces to the (Theorem 1, of [7]).

Remark 2.10. As a special case when S is non singular, then conditions $N(M/C)\subseteq N(M/A)$ and $N(M/B)\subseteq N(M/D)$ automatically hold and [by Theorem 1.4] $N[S/(M/D)]$ and $N[S/(M/B)]$ are non singular, further, $\rho\left[\tilde{S}\right]=\rho(M/C)+\rho(M/B)$. Hence it follows that each secondary Pivot transform of S is that the non singular. However we note that the non singularity of $\tilde{S}$ need not imply S is non singular.

Example 2.11. $A=\begin{bmatrix}1&0\\1&1\end{bmatrix}$, $B=\begin{bmatrix}1&1\\0&1\end{bmatrix}$, $C=\begin{bmatrix}1&1\\0&1\end{bmatrix}$, $D=\begin{bmatrix}1&0\\1&1\end{bmatrix}$

$$M=\left[\begin{array}{c|c}A&B\\\hline C&D\end{array}\right]\quad M=\left[\begin{array}{cc|cc}1&0&1&1\\1&1&0&1\\\hline 1&1&1&0\\0&1&1&1\end{array}\right].$$ Schur complement of $(M/A)=\begin{pmatrix}1&-1\\2&1\end{pmatrix}$,

$$(M/B)=\begin{pmatrix}1&2\\-1&1\end{pmatrix},(M/C)=\begin{pmatrix}1&2\\-1&1\end{pmatrix},(M/D)=\begin{pmatrix}1&-1\\2&1\end{pmatrix}.$$

$$S=\left[\begin{array}{c|c}(M/A)&(M/B)\\\hline (M/C)&(M/D)\end{array}\right],\quad S=\left[\begin{array}{cc|cc}1&-1&1&2\\2&1&-1&1\\\hline 1&2&1&-1\\-1&1&2&1\end{array}\right],\quad K=\left[\begin{array}{cc|cc}1&0&0&0\\0&1&0&0\\\hline 0&0&1&0\\0&0&0&1\end{array}\right]$$

$$V=\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix},\quad KVS=\left[\begin{array}{cc|cc}-1&1&2&1\\1&2&1&-1\\\hline 2&1&-1&1\\1&-1&1&2\end{array}\right],K_1V(M/C)=\begin{pmatrix}-1&1\\1&2\end{pmatrix}$$

$$K_1V(M/D)=[K_2V(M/A)]^T=\begin{pmatrix}2&1\\1&-1\end{pmatrix},$$

$K_2V(M/B)=\begin{pmatrix}-1&1\\1&2\end{pmatrix}$. Here $K_1V(M/C)$ and $K_2V(M/B)$ are non singular and $\left[S/(M/C)\right]=\begin{bmatrix}3&3\\0&3\end{bmatrix}$. $K_2V\left[S/(M/C)\right]=\begin{bmatrix}0&3\\3&3\end{bmatrix}$ is Con-EP$_2$. Therefore [S/(M/C)] is Con-s-k$_2$-EP$_2$.

$$\rho(KVS)=\rho(K_1V(M/C))+\rho(K_2V[S/(M/C)]).$$

That is $\rho(S)=\rho(M/C)+\rho[S/(M/C)]=3$. Since KVS is symmetric, KVS is Con-EP$_3$ which implies S is Con-s-k-EP$_3$. By (2.9).

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