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RESEARCH ARTICLE

ON CHARACTERIZATION OF A GENERALIZED (α,β) 'USEFUL' INFORMATION MEASURE

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ABSTRACT

In this communication a generalized measure of 'useful' information is defined which has a utility scheme and two probability distributions respectively. The recent development in information theory is described to the study of characterization results based on the purely functional equation approach. In this paper, a characterization theorem is proved here with help of a functional equation. A generalized (α,β) information measure of Ahmad and Khan (1977), the relative 'useful' information of jain and Tuteja (1986), the directed divergence of Rathie and Kannappan (1972), relative information of Kullback's (1959) and Pearson's χ^2 - statistic is a measure of discrepancy between the two distributions P and Q are special cases of the measure defined here.

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1. INTRODUCTION

Shannon's (1948) entropy measure

$$I_n(P) = -\sum_{i=1}^{n} p_i \log p_i \tag{1.1}$$

of the probability distribution

$$P = (p_1, p_2, \ldots, p_n),\ p_i > 0,\ \sum_{i=1}^{n} p_i = 1$$

was first generalized by Belis and Giuasu (1968) who defined 'useful' information as:

$$I_n(U;P) = \sum_{i=1}^{n} u_i p_i \log p_i \tag{1.2}$$

by attaching utility $u_i\,(>0)$ to the event with probability p_i. Followig the same idea Emptoz (1976) and Sharma et al. (1978) generalized the Havrda - Charvat (1967) entropy measure

$$I_n^\alpha(P) = \left(2^{1-\alpha}-1\right)^{-1}\left(\sum_{i=1}^{n} p_i^\alpha - 1\right), \alpha > 0 \tag{1.3}$$

through the introduction of utilities u_i to

$$I_n^\alpha(U;P) = \left(2^{1-\alpha}-1\right)^{-1}\sum_{i=1}^{n} u_i\left(p_i^\alpha - p\right), \alpha > 0 \tag{1.4}$$

to which function they also gave the name of 'useful' information.

2. Generalized (α,β) 'Useful' Information Measure

Consider the following two utility information schemes given by

$$S_n = [A, P, U, n], \tag{2.1}$$

where $A = (A_1, A_2, \ldots, A_n)$ is an experiment, $P = (p_1, p_2, \ldots, p_n), p_i > 0, \sum_{i=1}^{n} p_i = 1$ is the probability and $U = (u_1, u_2, \ldots, u_n), u_i > 0$ be the utility distribution of a set of n events after an experiment, an

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$$S_n = [A, Q, U, n], \tag{2.2}$$

where $Q = (q_1, q_2, \ldots, q_n), q_i > 0, \sum_{i=1}^{n} q_i = 1$ is the probability distribution of the same set A of an events after an experiment. In both the schemes (2.1) and (2.2) the utility distribution $U = (u_1, u_2, \ldots, u_n), u_i > 0$ be the same because it is assumed that the utility u_i of an outcome A is independent of its probability of occurrence p_i or predicted probability q_i [refer to Longo (1972)], then we obtain the measure

$$\begin{aligned} I_n^{\alpha,\beta}(U;P,Q) &= \left(2^{\beta-\alpha}-1\right)^{-1}\sum_{i=1}^{n} u_i p_i\left[\left(\frac{p_i}{q_i}\right)^{\beta-\alpha}-1\right], \quad \alpha, \beta > 0, \alpha \neq \beta \\ &= \left(2^{\beta-\alpha}-1\right)^{-1}\sum_{i=1}^{n} u_i\left(p_i^{\beta-\alpha+1} q_i^{\alpha-\beta} - p_i\right), \quad \alpha, \beta > 0, \alpha \neq \beta \end{aligned} \tag{2.3}$$

which we may call (2.3) a generalized 'useful' information measure in analogy with Belis and Guiasu (1968).

Particular Cases of the New Measure

(i)If we take $u_i = 1$ for each i in (2.3) we obtain the measure

$$\left(2^{\beta-\alpha}-1\right)^{-1}\sum_{i=1}^{n}\left(p_i^{\beta-\alpha+1} q_i^{\alpha-\beta} - p_i\right), \quad \alpha, \beta > 0, \alpha \neq \beta \tag{2.4}$$

which is (α,β) information measure studied by Ahmad and Khan (1997).

(ii) If we take $\alpha = 1$ in (2.3), we get the measure

$$I_n^{\alpha,\beta}(U;P,Q) = \left(2^{\beta-1}-1\right)^{-1}\sum_{i=1}^{n} u_i p_i\left[\left(\frac{p_i}{q_i}\right)^{\beta-1}-1\right], \quad \beta > 0, \beta \neq 1 \tag{2.5}$$

which is $\beta-$type relative 'useful' information obtained by Jain and Tuteja (1986).

(iii) If we take $u_i = 1$ for each i and $\alpha = 1$ in (2.3), we obtain a measure of directed divergence of Rathie and Kannappan (1972) which is given by:

$$I_n^{\beta}(P;Q) = \left(2^{\beta-1}-1\right)^{-1}\left(\sum_{i=1}^{n} p_i^{\beta} q_i^{1-\beta} - 1\right), \quad \beta > 0, \beta \neq 1 \tag{2.6}$$

(iv) If we take $u_i = 1$ for each i and $\alpha = 1$ and $\beta = 2$ in (2.3), we get

$$I_n^{1,2}(P,Q) = \left[\left(\sum_{i=1}^{n}\frac{p_i^2}{q_i}\right)-1\right] \tag{2.7}$$

which is Pearson's χ^2-statistic and is a measure of discrepancy between the two discrete populations P and Q.

(v) Further if $u_i = 1$, $\alpha = 1$ and $\beta \to 1$ in (2.3)

$$I(P;Q) = \sum_{i=1}^{n} p_i \log_2(p_i/q_i) \tag{2.8}$$

which is Kullback's (1959) measure of relative information that the distribution $P = (p_1, p_2, \ldots, p_n)$ provides about the distribution $Q = (q_1, q_2, \ldots, q_n)$. We may, therefore, take this measure as a generalized measure of the relative information that the distribution P provides about the distribution Q. If P is the distribution determined on the basis of an experiment, then this measure may be considered as a measure of the information on Q furnished by the experiment. So far various authors have described two different approaches to characterization results, with reference to Shannon's entropy. In the first, which we may call the purely axiomatic approach, one does not specify ab initio any particular form for the entropy function but obtains it directly from the listed properties. In the second approach one assumes a particular functional form for the entropy function and also lists some other properties on the basis of which a functional equation is derived whose solution is utilized to obtain the entropy function. In addition to these two approaches we also come across a third approach, particularly in relation to more general information measure, which we may call the purely functional approach. In this method one specifies the general form of the information measure involving one or more undetermined functions. The particular information measure is then shown to result from the solution of a functional equation involving the undetermined functions. The functional equation used does not necessarily arise from any basic properties of the information measure but is formulated without any reference to such properties. The recent development in information theory is described to the study of characterization results based on the purely functional equation approach. We shall characterize the measure $I_n^{\alpha,\beta}(U;P,Q)$. Application of $I_n^{\alpha,\beta}(P,Q)$ and $I_n^{\alpha,\beta}(U;P,Q)$ have been made recently in the theory of questionnaire (refer Picard 1972) and in the analysis of business and Accounting data (refer Sharma et al. (1976, 1978). Thus there arose a need of further studying statistical estimators of $I_n^{\alpha,\beta}(P,Q)$ and $I_n^{\alpha,\beta}(U;P,Q)$.

2. Characterization of $I_n^{\alpha,\beta}(U;PQ)$

We consider the following functional equation

$$F(uv, xy, st) = ux^{\beta-\alpha+1}s^{\alpha-\beta}F(v, y, t) + vyF(u, x, s) \tag{3.1}$$

for which we have the following lemmas.

Lemma 1: If the domain of F is $(0,\infty)\times(0,1)\times(0,1)$, the general solution of (3.1) is $F(u,x,u)=K\left(ux^{\beta-\alpha+1}s^{\alpha-\beta}-ux\right)$, where K is an arbitrary constant.

Proof : Interchanging u and v, x and y, and s and t in (3.1) we get

$$\begin{aligned}&F(vu,yx,ts)\\&=vy^{\beta-\alpha+1}t^{\alpha-\beta}F(u,x,s)+uxF(v,y,t)\end{aligned} \tag{3.2}$$

from (3.1) and (3.2) we obtain

$$\begin{aligned}&\left\{u\,x^{\beta-\alpha+1}-u\,x\right\}F\,(v,y,t\,)\\&=\left\{v\,y^{\beta-\alpha+1}-t^{\alpha-\beta}-v\,y\right\}F\,(u,x,s)\end{aligned} \tag{3.3}$$

Now, in the domain under consideration

$ux^{\beta-\alpha+1}s^{\alpha-\beta}-ux$ and $vy^{\beta-\alpha+1}t^{\alpha-\beta}-vy$ are not equal to zero. Hence (3.3) yields

$$\frac{F(u,x,s)}{ux^{\beta-\alpha+1}s^{\alpha-\beta}-ux}=\frac{F(v,y,t)}{vy^{\beta-\alpha+1}t^{\alpha-\beta}-vy}$$

which leads to

$$F(u,x,s)=K\left(ux^{\beta-\alpha+1}s^{\alpha-\beta}-ux\right) \tag{3.4}$$

where K is some constant. Substitution of (3.4) is indeed a solution (3.1), which proves the lemma.

Since, we propose to use functional equation (3.1) to characterize the 'useful' information measure (2.3) it is necessary to extend the domain of F to $[0,\infty)\times[0,1]\times[0,1]$.

We will now show that the solution (3.4) can be extended to the enlarged domain through continuity.

Lemma 2: If $\alpha\neq 1,\beta\neq\alpha$, then (3.4) is a solution of (3.1) in the domain $(0,\infty)\times(0,1)\times[0,1]$.

Proof : If $\alpha\neq 1,\beta\neq\alpha$, then $ux^{\beta-\alpha+1}s^{\alpha-\beta}-ux\neq 0$ in the new domain also, from which the lemma follows.

It therefore remains to consider the behavior of the equation (3.1) for

$$\left.\begin{matrix}(0,x,s)\\(0,0,s)\\(0,1,s)\end{matrix}\right\},\ x\in(0,1),s\in[0,1]$$

and,

$$\left.\begin{matrix}(u,0,s)\\(u,1,s)\end{matrix}\right\},u\in(0,\infty),s\in[0,1].$$

At all these points we define $F(u,x,s)$ by continuity to obtain

$$F(0,x,s)=\lim_{u\to 0}F(u,x,s)=0$$

$$F(0,0,s)=\lim_{\substack{u\to 0\\x\to 0}}F(u,x,s)=0$$

$$F(0,1,s)=\lim_{\substack{u\to 0\\x\to 1}}F(u,x,s)=0$$

$$F(u,0,s)=\lim_{x\to 0}F(u,x,s)=0$$

$$F(u,1,s)=\lim_{x\to 1}F(u,x,s)=K\left(us^{\alpha-\beta}-u\right)$$

It can be verified that these limiting values of $F(u,x,s)$ are solutions of (3.1). For example, taking $u=0,y=1,t=1,$ in (3.1) we get

$$F(0,x,s)=vF(0,x,s)$$

which implies $F(0,x,s)=0$, similarly, taking $x=y=1$ in (3.1) we get

$$F(uv,1,st)=us^{\alpha-\beta}F(v,1,t)+vF(u,1,s) \tag{3.5}$$

interchanging u and v, s and t, in (3.5) gives

$$F(vu,1,ts)=vt^{\alpha-\beta}F(u.1,s)+uF(v,1,t). \tag{3.6}$$

From (3.5) and (3.6), we obtain

$$\left(us^{\alpha-\beta}-u\right)F(v,1,t)=\left(vt^{\alpha-\beta}-v\right)F(u,1,s).$$

Here we have $u>0$; hence, if $s\neq 1$,

$$F(u,1,s)=A\left(us^{\alpha-\beta}-u\right)$$

where A is an arbitrary constant. For the case $s=1$, we get, putting $x=y=1,\ s=t=1$ in (3.1) we get

$$F\,(uv,1,1)=u\,F\,(v,1,1)+v\,F\,(u,1,1)$$

for which the solution is $F\,(u,1,1)=B\log u$, B being an arbitrary constant,

or $F\,(u,1,1)=0$.

The latter solution corresponds to the value

$$\lim_{x\to 0}F(u,x,1).$$

We summarize the above results as follows:

Iemma 3: If $\alpha \neq 1, \beta \neq \alpha$, the general continuous solution of (3.1) in the domain $[0,\infty)\times[0,1]\times[0,1]$ is given by (3.4).

We are now in a position to obtain a characterization theorem for $I_n^{\alpha,\beta}(U;P,Q)$

Theorem: A generalized 'useful' information measure is given by:

$$I_n^{\alpha,\beta}(U;P,Q)=\sum_{i=1}^{n}F(u_i;p_i,q_i) \tag{3.7}$$

where F satisfy the functional equation

(3.1) with the normalizing condition

$$F\left(1,1,\frac{1}{2}\right)=\left(2^{\beta-\alpha}-1\right)^{-1}\left(2^{\alpha-\beta}-1\right) \tag{3.8}$$

or

$$F\left(1,1,\frac{1}{2}\right)=1$$

Proof: From (3.8) and (3.4), we obtain

$$K\left[\left(\frac{1}{2}\right)^{\alpha-\beta}-1\right]=1$$

i.e., $K=\left(2^{\beta-\alpha}-1\right)^{-1}$.

Hence

$$\sum_{i=1}^{n}F(u_i;p_i,q_i)$$

$$=\left(2^{\beta-\alpha}-1\right)^{-1}\sum_{i=1}^{n}u_i\left(p_i^{\beta-\alpha+1}q_i^{\alpha-\beta}-p_i\right)$$

$$=I_n^{\alpha,\beta}(U;P,Q)$$

which concludes the proof of the theorem.

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