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RESEARCH ARTICLE

SOME CHARACTERIZATIONS OF S-α FUZZY SEMIGROUPS BY THEIR S-α LEVEL SUBGROUPS

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ABSTRACT

Some characterizations of S-α fuzzy semigroups by their S-α level subgroups are obtained. S-α fuzzy semigroups which have an identical family of S-α level subgroups are discussed. It is also derived a necessary and sufficient condition for two S-α fuzzy semigroups with an identical family of S-α level subgroups to be equal.
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INTRODUCTION

The notion of fuzzy set was introduced by Zadeh (1965). A.Rosenfeld (1971) defined fuzzy groups and many group theory results have been extended to fuzzy groups. The idea of level subgroups of a fuzzy group was initiated by Sivaramakrishna Das (1981). Prabir Bhattacharya (1987) analyzed the level subgroups of a fuzzy group in more detail and investigated whether the family of level subgroups of a fuzzy subgroup determine the fuzzy subgroup uniquely or not. Vasantha Kandasamy (2003) studied about Smarandache fuzzy semigroups. Sharma (2013) introduced the notion of α-fuzzy set, α-fuzzy group, α-fuzzy coset and determined their properties. Gowri and Rajeswari (2015) introduced the concept of S-α fuzzy semigroups, S-α fuzzy left, right cosets and S-α fuzzy normal subsemigroups and derived their characterizations. They have also introduced the concept of S-α level subgroup of an S-α fuzzy semigroup (Gowri and Rajeswari, 2016). In this paper, S-α fuzzy semigroups having an identical family of S-α level subgroups are investigated. It is also derived a necessary and sufficient condition for two S-α fuzzy semigroups with an identical family of S-α level subgroups to be equal. Throughout this paper, α will always denote a member of [0,1].

Definition 1.1. Let X be a non empty set. A **fuzzy subset** A of X is a function $A : X \to [0,1]$.

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Definition 1.2. A fuzzy subset A of a group G is called a **fuzzy subgroup** of G if

(i) $A(xy) \geq \min\{A(x), A(y)\}$
(ii) $A(x^{-1}) = A(x)$, for all $x, y \in G$.

Definition 1.3. Let A be a fuzzy subset of a set S. For $t \in [0,1]$, the set $A_t = \{x \in S / A(x) \geq t\}$ is called a **level subset** of A.

Definition 1.4. Let G be a group and A be a fuzzy subgroup of G. The subgroups A_t, $t \in [0,1]$ and $t \leq A(e)$, are called **level subgroups** of A.

Definition 1.5. A semigroup S is said to be a **Smarandache semigroup (S-semigroup)** if there exists a proper subset P of S which is a group under the same binary operation in S.

Definition 1.6. Let A be a fuzzy subset of a group G. Let $\alpha \in [0,1]$. Then an **α-fuzzy subset of G(with respect to a fuzzy set A)**, denoted by A^{α}, is defined as $A^{\alpha}(x) = min\{A(x), \alpha\}$, for all $x \in G$.

Definition 1.7. Let G be an S-semigroup. Let A be a fuzzy subset of G and let $\alpha \in [0,1]$. A is called a **Smarandache-α fuzzy semigroup (S-α fuzzy semigroup)** if there exists a proper subset P of G which is a group and the restriction of A to P ($A_P: P \to [0,1]$) is such that ${A_P}^{\alpha}$ is a fuzzy group.

Definition 1.8. Let G be an S-semigroup. Let $A: G \to [0,1]$ be a fuzzy subset of G. For $t \in [0,1]$, a **Smarandache-α level**

subset (S-α level subset) of the fuzzy subset A, denoted by ${A_{P_t}}^{\alpha}$, is defined as ${A_{P_t}}^{\alpha} = \{x \in P / {A_P}^{\alpha}(x) \geq t\}$, where P is a proper subset of G which is a group.

Definition 1.9. Let A be an S-α fuzzy semigroup of an S-semigroup G relative to a group P in G. If $t \in [0,1]$ and $t \leq {A_P}^{\alpha}(e)$, then the subgroups ${A_{P_t}}^{\alpha}$ are said to be **Smarandache-α level subgroups (S-α level subgroups)** of A with respect to P.

Result 1.10[11]. Let A be an S-α fuzzy semigroup of an S-semigroup G relative to a finite group P. Let $Im({A_P}^{\alpha}) = \{t_i / {A_P}^{\alpha}(x) = t_i$, for some $x \in P\}$. Then ${A_{P_{t_i}}}^{\alpha}$'s are the only S-α level subgroups of A with respect to P.

Result 1.11[11] Let G be an S-semigroup. Then any subgroup H of a group P in G can be realised as an S-α level subgroup of some S-α fuzzy semigroup of G relative to P.

Some Characterizations of S-α fuzzy semigroups by their S-α level subgroups

In this section, S-α fuzzy semigroups having an identical family of S-α level subgroups are analysed. It is also derived a necessary and sufficient condition for two S-α fuzzy semigroups with an identical family of S-α level subgroups to be equal. A relation is defined on the set of all S-α fuzzy semigroups and it is also investigated.

Theorem 2.1. Let G be an S-semigroup and A be an S-α fuzzy semigroup of G relative to a finite group P in G. If t_i and t_j are in $Im\ ({A_P}^{\alpha})$ such that ${A_{P_{t_i}}}^{\alpha} = {A_{P_{t_j}}}^{\alpha}$ then $t_i = t_j$.

Proof

Let x_i be a preimage of t_i under ${A_P}^{\alpha}$. Then ${A_P}^{\alpha}(x_i) = t_i$ which implies that $x_i \in {A_{P_{t_j}}}^{\alpha}$, since ${A_{P_{t_i}}}^{\alpha} = {A_{P_{t_j}}}^{\alpha}$. Therefore ${A_P}^{\alpha}(x_i) \geq t_j$ which implies that $t_i \geq t_j$. Similarly it is easy to see that $t_j \geq t_i$ and hence $t_i = t_j$.

Remark 2.2. Two S-α fuzzy semigroups of an S-semigroup G relative to a finite group P in G may have an identical family of S-α level subgroups with respect to P, but the S-α fuzzy semigroups need not be equal. This is illustrated in the following example. Let $G = \{e, a, b, c, d, e, f, g\}$ which is a semigroup by the following table.

*	e	a	b	c	d	f	g
e	e	a	b	c	d	f	g
a	a	e	c	b	a	a	a
b	b	c	e	a	b	b	b
c	c	b	a	e	c	c	c
d	d	a	a	a	d	f	g
f	f	b	b	b	d	f	g
g	g	c	c	c	d	f	g

Let $P = \{e, a, b, c\}$ which is the klein four group and hence G is an S-semigroup.

Let $A: G \to [0,1]$ be defined as $A(x) = \begin{cases} t_0, & \text{if } x = e \\ t_1, & \text{if } x = a, b \\ t_2, & \text{if } x = c \\ 0, & \text{otherwise} \end{cases}$

where $t_i \in [0,1]$, $0 \leq i \leq 2$ and $t_0 > t_1 > t_2$. Choose $\alpha \geq t_0$. Then for $x \in P$, ${A_P}^{\alpha}(x) = min\{A_P(x), \alpha\} = A(x)$. Clearly ${A_P}^{\alpha}$ is a fuzzy group and hence A is an S-α fuzzy semigroup relative to P. Also $Im\ ({A_P}^{\alpha}) = \{t_0, t_1, t_2\}$. Then the only S-α level subgroups of A with respect to P are ${A_{P_{t_0}}}^{\alpha} = \{e\}$, ${A_{P_{t_1}}}^{\alpha} = \{a, b, e\}$ and ${A_{P_{t_2}}}^{\alpha} = P$. Now we take $s_0, s_1, s_2 \in [0,1]$ such that $s_0 > s_1 > s_2$ and $\{t_i\} \cap \{s_i\} = \emptyset$

Let $B: G \to [0,1]$ be defined as $B(x) = \begin{cases} s_0, & \text{if } x = e \\ s_1, & \text{if } x = a, b \\ s_2, & \text{if } x = c \\ 0, & \text{otherwise} \end{cases}$

Let $\alpha \geq s_0$. Then ${B_P}^{\alpha}$ is a fuzzy group and hence B is an S-α fuzzy semigroup relative to P. Thus the only S-α level subgroups of B with respect to P are ${B_{P_{s_0}}}^{\alpha} = \{e\}$, ${B_{P_{s_1}}}^{\alpha} = \{a, b, e\}$ and ${B_{P_{s_2}}}^{\alpha} = P$. Thus A and B have the same family of S-α level subgroups relative to P and it is clear that A is not equal to B.

Theorem 2.3. Let A and B be two S-α fuzzy semigroups of an S-semigroup G relative to a finite group $P \subset G$. Assume that A and B have identical family of S-α level subgroups with respect to P. If $Im({A_P}^{\alpha}) = \{t_0, t_1, \ldots, t_r\}$ and $Im({B_P}^{\alpha}) = \{s_0, s_1, \ldots, s_k\}$, where $t_0 > t_1 > \ldots > t_r$ and $s_0 > s_1 > \cdots > s_k$, then

(i) $r = k$

(ii) ${A_{P_{t_i}}}^{\alpha} = {B_{P_{s_i}}}^{\alpha}$, $0 \leq i \leq r$

(iii) if $x \epsilon P$ such that ${A_P}^{\alpha}(x) = t_i$, then ${B_P}^{\alpha}(x) = s_i$, $0 \leq i \leq r$.

Proof

Since $Im({A_P}^{\alpha}) = \{t_0, t_1, \ldots, t_r\}$ and $Im({B_P}^{\alpha}) = \{s_0, s_1, \ldots, s_k\}$, the only S-α level subgroups of A and B with respect to P are the two families $\left\{{A_{P_{t_0}}}^{\alpha}, {A_{P_{t_1}}}^{\alpha}, \ldots, {A_{P_{t_r}}}^{\alpha}\right\}$ and $\left\{{B_{P_{s_0}}}^{\alpha}, {B_{P_{s_1}}}^{\alpha}, \ldots, {B_{P_{s_k}}}^{\alpha}\right\}$ respectively (by theorem 1.10). But by assumption, $\left\{{A_{P_{t_0}}}^{\alpha}, {A_{P_{t_1}}}^{\alpha}, \ldots, {A_{P_{t_r}}}^{\alpha}\right\} = \left\{{B_{P_{s_0}}}^{\alpha}, {B_{P_{s_1}}}^{\alpha}, \ldots, {B_{P_{s_k}}}^{\alpha}\right\}$. Therefore r must be equal to k which proves (i).
(ii) Since $t_0 > t_1 > \ldots > t_r$ and $s_0 > s_1 > \cdots > s_k$, by [10, remark 2.13], we have the following two chains of S-α level subgroups of A and B with respect to P:
$\{e\} = {A_{P_{t_0}}}^{\alpha} < {A_{P_{t_1}}}^{\alpha} < \ldots < {A_{P_{t_r}}}^{\alpha} = P$ and ${B_{P_{s_0}}}^{\alpha} < {B_{P_{s_1}}}^{\alpha} < \ldots < {B_{P_{s_r}}}^{\alpha} = \text{P}$. Therefore, if $t_i < t_j$, then ${A_{P_{t_i}}}^{\alpha} > {A_{P_{t_j}}}^{\alpha}$ and if $s_i < s_j$, then ${B_{P_{s_i}}}^{\alpha} > {B_{P_{s_j}}}^{\alpha}$. Clearly, ${A_{P_{t_0}}}^{\alpha} = \{e\} = {B_{P_{s_0}}}^{\alpha}$. By assumption, ${A_{P_{t_1}}}^{\alpha} = {B_{P_{s_j}}}^{\alpha}$, for some $j > 0$. Suppose that $j > 1$. Then $s_1 > s_j$ which implies that ${B_{P_{s_1}}}^{\alpha} < {B_{P_{s_j}}}^{\alpha}$. Also ${B_{P_{s_1}}}^{\alpha} = {A_{P_{t_i}}}^{\alpha}$, for some $i > 1$. Therefore ${A_{P_{t_i}}}^{\alpha} < {B_{P_{s_j}}}^{\alpha}$. Now ${B_{P_{s_j}}}^{\alpha} = {A_{P_{t_1}}}^{\alpha} < {A_{P_{t_i}}}^{\alpha}$, since $i > 1$. This implies that ${B_{P_{s_j}}}^{\alpha} < {A_{P_{t_i}}}^{\alpha}$ which contradicts ${A_{P_{t_i}}}^{\alpha} < {B_{P_{s_j}}}^{\alpha}$. Therefore ${A_{P_{t_1}}}^{\alpha} = {B_{P_{s_1}}}^{\alpha}$. Similarly, it can be easily proved that ${A_{P_{t_i}}}^{\alpha} = {B_{P_{s_i}}}^{\alpha}$, where $i = 2\ to\ r$.
(iii) Suppose that $x \in P$ such that ${A_P}^{\alpha}(x) = t_i$ and ${B_P}^{\alpha}(x) = s_j$. Then $x \in {A_{P_{t_i}}}^{\alpha}$ and $x \in {B_{P_{s_j}}}^{\alpha}$. By (ii), ${A_{P_{t_i}}}^{\alpha} = {B_{P_{s_i}}}^{\alpha}$ which implies that ${B_P}^{\alpha}(x) \geq s_i$. Therefore $s_j \geq s_i$ which

implies ${B_{P_{s_j}}}^{\alpha} \leq {B_{P_{s_i}}}^{\alpha}$. Moreover, by (ii), ${B_{P_{s_j}}}^{\alpha} = {A_{P_{t_j}}}^{\alpha}$ which gives ${A_P}^{\alpha}(x) \geq t_j$. This implies that $t_i \geq t_j$ and hence ${A_{P_{t_i}}}^{\alpha} < {A_{P_{t_j}}}^{\alpha}$. Now by (ii), ${B_{P_{s_i}}}^{\alpha} < {B_{P_{s_j}}}^{\alpha}$. Therefore ${B_{P_{s_i}}}^{\alpha} = {B_{P_{s_j}}}^{\alpha}$. By theorem 2.1, $s_i = s_j$ and hence ${B_P}^{\alpha}(x) = s_i$.

Theorem 2.4. Let A and B be two S-α fuzzy semigroups of an S-semigroup G relative to a finite group $P \subset G$. Let A and B have identical families of S-α level subgroups with respect to P. Then ${A_P}^{\alpha} = {B_P}^{\alpha}$ if and only if $Im\ ({A_P}^{\alpha}) = Im\ ({B_P}^{\alpha})$.

Proof

Suppose that $Im\ ({A_P}^{\alpha}) = Im\ ({B_P}^{\alpha})$. Let $Im({A_P}^{\alpha}) = \{t_0, t_1, \dots, t_r\}\ and\ Im({B_P}^{\alpha}) = \{s_0, s_1, \dots, s_r\}$, where $t_0 > t_1 > \dots > t_r$ and $s_0 > s_1 > \cdots > s_r$. Since $s_0 \in Im\ ({B_P}^{\alpha})$, $s_0 \in Im\ ({A_P}^{\alpha})$ which implies $s_0 = t_{k_0}$ for some k_0. If $t_{k_0} \neq t_0$, then $t_{k_0} < t_0$. $s_0 \in Im\ ({A_P}^{\alpha}) \Rightarrow s_1 = t_{k_1}$ for some k_1. Also $s_0 > s_1 \Rightarrow t_{k_0} > t_{k_1}$. If we proceed in this way, we have $t_{k_0} > t_{k_1} > \cdots > t_{k_r}$, where $t_{k_0} < t_0$ which contradicts $Im\ ({A_P}^{\alpha}) = Im\ ({B_P}^{\alpha})$. Therefore $t_{k_0} = t_0$ and hence $s_0 = t_0$. A similar argument will lead to $s_i = t_i$, $0 \leq i \leq r$. Now let $x \in P$. Then ${A_P}^{\alpha}(x) = t_k$ for some k. Using theorem 2.3, ${B_P}^{\alpha}(x) = s_k$ which implies that ${A_P}^{\alpha}(x) = {B_P}^{\alpha}(x)$, since $s_i = t_i$, for all i. Therefore ${A_P}^{\alpha} = {B_P}^{\alpha}$. Converse part is trivial.

Definition 2.5. Let G be an S-semigroup and ${F_P}^{\alpha}$ denotes the set of all S-α fuzzy semigroups of G relative to a finite group P in G. For $A, B \in {F_P}^{\alpha}$, we define $A \sim B$ if and only if A and B have the same family of S-α level subgroups of G with respect to P.

Remark 2.6. By remark 2.2, two elements A and B in ${F_P}^{\alpha}$ may satisfy the condition $A \sim B$, but they need not be equal.

Theorem 2.7. The relation $\sim$, defined in 2.5, is an equivalence relation.

Proof

It is obvious that $\sim$ is reflexive and symmetric. If $A \sim B$ and $B \sim C$, then clearly A and C will have the same family of S-α level subgroups and hence $A \sim C$.

Corollary 2.8. If G is an S-semigroup, then the number of distinct equivalence classes in ${F_P}^{\alpha}$, under the relation $\sim$ defined in 2.5, is finite.

Proof

Since P is finite and each S-α level subgroup with respect to P is a subgroup of P, the number of S-α level subgroups with respect to P is finite. By theorem 1.11, any subgroup of P can be realised as an S-α level subgroup of some S-α fuzzy semigroups of G relative to P. Thus it follows that the number of possible chains of S-α level subgroups is also finite. Since each equivalence class is characterized completely by its chain of S-α level subgroups with respect to P, the number of distinct equivalence classes in ${F_P}^{\alpha}$ is finite.

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