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RESEARCH ARTICLE

FIXED POINT THEOREMS WITH BEST APPROXIMATION FLAVOR

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ABSTRACT

In this paper, we present some fixed point theorems with best approximation flavor. Our results generalize the results of (8) by relaxing the compactness condition of the set X. The original result in this direction was due to (1) for a continuous map defined on a compact convex set.

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INTRODUCTION

Fixed point theorems are very important tools for providing evidence of the existence and uniqueness of solutions to various mathematical models. Fixed point theory focuses on the strategies for solving non-linear equations of the kind T x = x in which T is a self-mapping defined on a subset of a metric space, a normed linear space, a topological vector space or some pertinent framework. But when T is not a self-mapping, it is possible that T x = x has no solution. Subsequently, one targets to determine an element x that is in some sense close proximity to T x. In fact, best approximation theorems and best proximity point theorems are suitable to be explored in this direction. A well-known best approximation theorems, due to Fan (1969), ascertains that if K is a non-empty compact convex subset of a Hausdorff locally convex topological vector space E and T : K $\rightarrow$ E is a continuous non-self mapping , then there exists an element x in such a way that d(x, Tx) = d (Tx , K). Several authors, including Prolla (1982), Reich (1978) and Sehgal and Singh (1988, 1989), have accomplished extensions of this theorem in various directions. Moreover, a result that unifies all such best approximation theorems has been obtained by Vetrivel *et al*. (1992). Let E be a Linear Hausdorff topological vector space and E* the dual of E.

Let H, X be non-empty subsets of E, we put

$B_H X = \bar{X} \cap \overline{H \quad X}$ and $I_H X = X \cap (B_H X)^C$

where A is the closure of A⊂E and A^C is the compliment of A.$B_H X$ and $I_H X$ are boundary and interior respectively of X relative to H.

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Main Results

Relaxing the compactness of X in Theorem 5 of Takahashi(8), we prove the following

Theorem 2.1 Let X be a non-empty weakly compact convex subset of a Linear Hausdorff topological vector space E and T be a continuous mapping of X into E. Then, either there exist $y_0 \in X$ such that y_0 and Ty_0 can not be separated by a continuous linear functional, or there exist $x_0 \in X$ and $g \in E$ such that

$$g(x_0 \quad Tx_0) < 0 \leq \inf_{y \in X} g(x_0 \quad y).$$

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In order to prove if we need the following Lemma (2.1) for a topological vector space. It was proved by Ganguly and Wadhwa (1994) in Theorem 2 for a normed Linear space, based upon the results of Sehgal, Singh and Whitfield (1992).

Lemma 2.1 Let X be a non-empty, weakly compact subset of a Hausdorff topological vector space E and Let F be a real valued function on $X \times X$ satisfying:

(i) For each $y \in X$, the function $F(x, y)$ of X is upper semi continuous;
(ii) For each $x \in X$, the function $F(x, y)$ of y is convex;
(iii)$F(x, x) \geq C$for every $x \in X$ for some real C.

Then, there exists $x_0 \in X$ such that $F(x_0, y) \geq C$ for all $y \in X$.

Proof of Theorem 2.1. Suppose that for each $x \in X$, there exists $f \in E$ such that $f(x \quad Tx) < 0$. Setting $A_f = \{x \in X: f(x \quad Tx) < 0\}$ for each$f \in E$,we have
X = $_{f \in E} A_f$. Since X is weakly compact, there exist a finite family $\{f_1, f_2, \ldots\ldots, f_n\}$ in E* such that $X = {}_{i=1}^{n} A_{f_i}$. Let $\{\beta_1, \beta_2, \ldots\ldots, \beta_n\}$ be a partition of unity corresponding to this covering $\{A_{f_i}\}$ of X. Define a real valued function F on $X \times X$ by setting.

$F(x, y) = \sum_{i=1}^{n} \beta_i(x) f_i(x \quad y)$. Then by Lemma 2.1,

there exists $x_0 \in X$ such that

$F(x_0, y) = \sum_{i=1}^{n} \beta_i(x_0) f_i(x_0 \quad y) \geq 0$for all $y \in X$.

On the other hand, we know that

$F(x_0, x_0) = \sum_{i=1}^{n} \beta_i(x_0) f_i(x_0 \quad Tx_0) < 0$.

By putting,$g = \sum_{i=1}^{n} \beta_i(x_0) f_i$, we complete the proof.

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As direct consequences of Theorem 2.1, we have the following two theorems.

Theorem 2.2 Let X be a non-empty weakly compact convex subset of a locally convex topological vector space E and T be a continuous mapping of X into E. If for each $x \in X$, there exists $x_1 \in X$and $\lambda \geq 0$ such that $Tx \quad x = \lambda(x_1 \quad x)$, then T has a fixed point.

Proof. Suppose T has no fixed point. By Theorem 2.1, there exists $x_0 \in X$ and $g \in E$ such that

$$g(x_0 \quad Tx_0) < 0 \leq \inf_{y \in X} g(x_0 \quad y).$$

For this x_0, we can choose $x_1 \in X$ and $\lambda \geq 0$ such that

$Tx_0 \quad x_0 = \lambda(x_1 \quad x_0)$. Since T has no fixed point, $\lambda > 0$.

Hence we have

$g(x_0 \quad Tx_0) < 0 \leq \frac{1}{\lambda} g(x_0 \quad Tx_0)$, a contradiction,

therefore, we have a fixed point.

Theorem 2.3 Let H be a closed convex subset of a locally convex topological vector space E and T be a continuous mapping of H into H. If there exists a weakly compact convex subset X of H such that for each$x \in B_H X$, there exists $x_1 \in X$ and $\lambda \geq 0$ with $Tx \quad x = \lambda(x_1 \quad x)$, then T has a fixed point in H.

Proof. Consider the restriction to X of T. If T has no fixed point in X, by Theorem 2.1, there exists $x_0 \in X$ and $g \in E$ such that

$$g(x_0 \quad Tx_0) < 0 \leq \inf_{y \in X} g(x_0 \quad y).$$

Let$x_0 \in I_H X$. Since$Tx_0 \in H$, we can choose$\lambda (0 < \lambda < 1)$ small enough is that $y = \lambda T x_0 + (1 \quad \lambda) x_0$ lies in X.Hence we obtain

$g(x_0 \quad Tx_0) < 0 \leq \lambda g(x_0 \quad Tx_0)$, a contradiction.

Similarly, we obtain a contradiction for the case of $x_0 \in B_H X$. Therefore, T has a fixed point.
For a normed vector space, we have from Lemma 2.1.

Theorem 2.4 Let X be a non-empty weakly compact convex subset of a normed linear space E and T be a continuous mapping of X into E. Then, there exists$x_0 \in E$such that

$$\min_{y \in X} \|Tx_0 \quad y\| \geq \min_{x \in X} \|Tx \quad x\|$$

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Proof. Define a real valued function F on $X \times X$by$F(x, y) = \|Tx \quad y\|$. The result follows from Lemma 2.1.

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