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RESEARCH ARTICLE

ON THE PHASE TYPE DISTRIBUTION UNDER TWO SIDED RISK RENEWAL PROCESS

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ABSTRACT

This paper is concerned with the ruin probability of two sided jump renewal problems under phase type claim/arrival distribution. We propose a class of finite dimensional phase type distributions as models for claim distribution .Different forms of phase type distributions are also discussed .Lindley distribution is approximated for claim distribution and a modified inter arrival claim distribution is also presented. A kind of ruin components is also given in the process.

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INTRODUCTION

Phase type distributions have been applied in large scale in risk theory ever since its introduction by Neuts (1981) and (1975). Neuts introduced the probability distributions of phase type in 1975 of which the Erlang and hyper exponential distributions are of special type. Here emphasis the use of phase type distributions in risk theory and at the same time to outline a recent line of research which includes statistical inference for phase type distributions and related measures such as ruin probability, deficit at ruin. For more, we refer to Neuts (1995), Asmussen (2003), O'clinncide (1999), Asmussen and Bladt (1996) generalises risk models to situations with Markov modulated arrivals and to situations where the premium depends on the current reserve. Asmussen *et al*. (2002) developed an algorithmic solution to the finite time horizon ruin probability. Statistical inference for phase type distributions is of more recent date where likelihood estimation was first proposed by Asmussen *et al*. (2002) using the EM-algorithm whereas a Markovian claim Monte Carlo(MCMC) based approach was suggested by Bladt *et al*. (2003). Recently many works have been reported by using discrete/continuous phase-type distribution for inter arrival time distribution Hu Yang & Zhimin Zang (2010) risk models with phase type claims have been considered by many researchers. Gerber *et al*. (1987), Derik *et al*. (2004) and Stanford *et al*. (1994) considered the time to ruin or its Laplace transforms for the renewal risk model, where both the claim inter arrival time distribution and the claim size distribution are phase type. In this paper we shall assume an arbitrary interclaim distribution k(t) and phase type distributed individual claim sizes. After giving a brief review of properties of phase type distributions, different types of phase type distributions and two sided jump risk renewal process, we shall consider the ruin probability and other measures within the rest of the paper.

Phase type distribution

Consider a stochastic process $\{X(t)\}, t \geq 0$ with state space S. $\{X(t)\}$ is a continuous markov chain if it is characterised by the Markov property,

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$P[X(t_{k+1}) = x_{k+1} / X(t_k) = x_k, ..., X(t_0) = x_0] = P[X(t_{k+1}) = x_{k+1} / X(t_k) = x_k]$ for any $0 \le t_0 \le t_1 \le ...... \le t_k \le t_{k+1}$ and $x_i \in S$ .We assume that the state S of the continuous time absorbing Markov process $\{X(t)\}$ is finite and contains the set of transient states $S_T = \{1, 2, ..., n\}$ and a single absorbing state n+1.Then $\{X(t)\}$ has an intensity matrix of the form $Q = \begin{bmatrix} T & t \\ 0 & o \end{bmatrix}$ where T is $n \times n$ dimensional matrix describing only transitions between transient states. The $n \times 1$ vector t contains transition intensities fron transient state to the absorbing state, the row vector 0 consists entirely of 0's since no transition from the absorbing state to transition states can occur .The remaining element of the matrix Q is o which gives the transition rate off the absorbing state. Since the intensities of rows must sum to zero, we notice that $t = -Te$ where e=(1,1,…,1)'.

Let $®_i = P[X_0 = i], i = 1,...,n, P[X_0 = n+1] = 0$ denote the initial probabilities. Let$®_i = [®_1, ®_2, ..., ®_n]$ denote the initial distribution of $\{X(t)\}_{t \ge 0}$ over the transient states only. The transient states are called phases. Now, A phase type distribution (PHD) is defined as the distribution of the lifetime X, ie, the time to enter an absorbing state from the set of transient states S of an absorbing continuous time Markov process $\{X(t)\}_{t \ge 0}$ .The time until absorption

$² = \inf\{t \ge 0 / X_t = n+1\}$ is said to have a phase type distribution and we write $² \sim PH(®, T)$.

Properties of PHD

We recall that the matrix exponential e^Q is defined by the power series expansion $\sum_{k \ge 0} \frac{Q^k}{k!}$. The distribution function of a phase type distributed variable with representation $(®, T)$ is given by $F(x) = 1 - ® e^{Tx} e, x \ge 0$ and its associated density function is given by $f(x) = ® e^{Tx} t \ , x \ge 0$ .Assume that Markov process $\{X(t)\}_{t \ge 0}$ with an infinitesimal generator Q is associated with a random variable X. The transition matrix P_s contains elements $P_s(i, j) = P[X_s = j / X_0 = i]$ which is the probability of being in phase j at time S, given that the initial phase is i. These probabilities are given by $P_S = e^{QS}$ where $e^{QS} = \begin{bmatrix} e^{TS} & e - e^{TS} e \\ 0 & 1 \end{bmatrix}$.

The expected total time spent in phase j before absorption, given that the initial phase is i equals $-T^{-1}(i, j)$ .The ith moment of a PHD is derived from $«_i = E(X^i) = (-i)^i i! ® T^{-k} e$.

Log PH distribution

The Log PH distribution ,denoted by Log PH (π,T) was introduced by Ramasami (2011), and is defined as the distribution of the random variable Y=exp(X) where X has a PH distribution with (π,T) i.e if Y is Log PH distributed then log (Y) is a PH distributed random variable. The distribution function and density function is derived as

$$F_Y(y) = P(Y \le y)$$

$$= 1 - ® e^{T \log Y} e, y \ge 1$$

and $f_Y(y) = \frac{1}{y} ® e^{T \log y} t, y \ge 1, t = -Te$

Exponentiated PH distribution

Here we introduce Exponentiated PH distribution (π,T),denoted by ExpPH (π,T), ,defined as the distribution of the random variable Z=log (X) where X has a PH distribution with (π,T) i.e if Z is an ExpPH distributed then exp(Z) is a PH distributed random variable. The distribution function and density function could be derived as

$$F_Z(z) = 1 - ® e^{Te^z} e, -\infty \le z \le \infty$$

and $f_Z(z) = ® e^{Te^z + z} t, -\infty \le z \le \infty$

we also note that distribution of Exponent of ExpPH is PH distribution.

Moment Generating Function

We consider the mgf of the Exponentiated PH distribution in this section .We use the definition of mgf ; for $-\infty \leq r \leq \infty$,

$$E(e^{rz}) = \int_{-\infty}^{\infty} e^{rz} ® e^{Te^{z}} e^{z} t dz$$

$$E(e^{rz}) = ® T^{-r} t \int_{0}^{\infty} u^{r} e^{u} T^{-1} du$$

Using the transformation,

$$E(e^{rz}) = ® T^{-r} (-1) r! e$$

Also the probability generating function,

$$P(t) = -® e(T^{-1} \log u)!$$

Inverted PH distribution

Here we introduce one more PH distribution called Inverted PH distribution (π,T), denoted by Inverted PH (π,T),defined as the distribution of the random variable $Q = X^{-1}$ where X has a PH distribution with (π,T) i.e if Q is an Inverted PH (π,T), distributed then Q^{-1} is a PH distributed random variable. The distribution function and density function could be derived as,

$$F_{Q}(q) = ® e^{T/q} e, q > 0$$

and $f_{Q}(q) = ® e^{T/q} \dfrac{t}{q^{2}}, q > 0$,noting that the distribution of inverted value of Inverted PH is PH distribution itself.

Moments

In this section we consider the moments of Inverted PH distribution .By definition of the k th is given by

$$E(q^{k}) = \int_{0}^{\infty} q^{k} e^{T/q} ® t q^{-2} dq$$

By transformation;

$$E(q^{k}) = ® t \int_{0}^{\infty} \left(T/u\right)^{k} e^{u} T^{-1} du$$

i.e, $E(\mathrm{q}^{k}) = -T^{k} ® e(-k)!$ .Mean and variance are $E(q) = T ® e$ and

$V(q) = T^{2} e(2®^{2} - ®)$, by Kurba's classical case

Two sided jump renewal process

We now consider an immediate application of the phase type renewal theory to the following model; Let $R(t)$ be the surplus process, then

Following the idea of Dong and Liu (2013) we consider the surplus process as

$$R(t) = u + pt + \sum_{i=1}^{M(t)} X_{i} - \sum_{j=1}^{N(t)} Y_{j} \quad ,t \geq 0 \tag{1}$$

Where u≥0,the initial surplus, p>0,the constant premium rate , $\sum_{i=1}^{M(t)} X_{i}$ Compound Poisson process with intensity λ representing the total random gain (premium income or investment or annuity) until time t. X_i's are independent and identically distributed random variables with common density f and mean μ_x . Here we assume that X follows an Erlangian (2, β) and Laplace transforms of f be $\hat{f}(s)=\int_0^{\infty} e^{-sx} f(x)dx$ whereas $\sum_{j=1}^{N(t)} Y_j$ is an aggregate claims N(t) is a Renewal process denoting the number of claims up to time t with interclaim { H_i}.The H_i's are i.i.d random variables with common density g and Laplace transform $\hat{g}(s)=\int_0^{\infty} e^{-sh} g(h)dh$. The claim sizes {$Y_j$} are positive random variables with a common distribution function Q,density q and mean μ_y. Laplace transform $\hat{q}(s)=\int_0^{\infty} e^{-sy} q(y)dy$ .Here we assume that Y follows Lindley distribution with parameter θ.

$$q(y)=\frac{-^2}{1+-}(1+y)e^{-^- y}, ^- >0$$ (2008).

and $\hat{q}(s)=\dfrac{-^2(1+s+^-)}{(1+^-)(s+^-)^2}$. Also we assume that $\{X_i\},\{Y_j\},\{M(t)\}$ and {N(t)} are mutually independent and $(p+ª«_x e).E(H)>«_y$ for ensuring a positive security loading condition.

Let $T=\inf\{t\geq 0, \mathrm{R(t)}<0\}$ be the ruin time, R (T_) be the surplus immediately before ruin and $|R(T)|$ be the deficit at ruin, again we define the probability of ruin $\cdot\ (u)=P[T<\infty / R(0)=0)$,u≥0.

Again, Let $T=\sum_{i=1}^{n} H_i$ be the time when n^{th} claim occurs,T_0=0.Since ruin only occurs at the epochs where claims occur, then we define the discrete time process $\tilde{R}=\{\tilde{R}_n, n=0,1,2...\}$ and $\tilde{R}_0=0$

Again $\tilde{R}n=R(T_n)$ denotes the surplus immediately after the n^{th} claim.

$$\text{Now, } \tilde{R}_n=u+pT_n+\sum_{i=1}^{M(\mathrm{T}_n)} X_i-\sum_{k=1}^{n} Y_k$$

$=u+p\tilde{T}_n-\sum_{k=1}^{n} Y_k$ where $\tilde{T}_n=T_n+\sum_{i=1}^{M(T_n)} X_i / p$ with $\tilde{T}_0=0$ which corresponds to the Sparre Anderson risk model (1957).

$\bar{R}(t)=u+pt-\sum_{i=1}^{\bar{N}(t)} Y_i$ where the initial surplus u and the claim size Y_i are exactly the same as those in model (1).The counting number process $\bar{N}(t)$ denotes the number of claims up to time t with the modified interclaim times $Z_i=\tilde{T}_i-\tilde{T}_{i-1}$ .Clearly Z_i are i.i.d random variables with a common density k(t).Following the idea of Rebello & Thampi (2017), k(t) can be taken as a linear combination of exponential distribution as

$k(t)=a_1 e^{-R_1 t}+a_2 e^{-R_2 t}+a_3 e^{-R_3 t}$ where a_1, a_2, a_3, R_1, R_2 & R_3 are suitably chosen constants.

Ruin Components

We are interested in calculating the probability of ruin for an infinite time horizon

$\cdot\ (u)=P[\inf_{0\leq t<\infty} R(\mathrm{t})<0 / \mathrm{R}_0=u]$ .Let M ,maximum aggregate loss of {R(t)},by concatenating the ascending ladder height of $\{R(t)\}$,we have the terminating Markov process $\{m_x\}$ with a defective initial vector $®_+$ and transition matrix $T+t®_+$ which results in the ladder height distribution as $PH(®_+,T)$ and maximal aggregate loss M is phase type with representation $(®_+, T+t®_+)$, see Asmussen (2000), and $®_+=®\hat{k}(T+t®_+)$ where $\hat{k}(T+t®_+)$, probability generating function, is equal to $\int_0^{\infty}(T+t®_+)^t k(t)dt$ where k(t) is the pdf of the interclaim time Z_i $\therefore ®_+=®\int_0^{\infty}(T+t®_+)^t \sum_{i=1}^{3} a_i e^{-R_i t}dt$ .Now the ruin probability $\cdot\ (u) = P[M > u]$ where $M \sim PH(®_{+}, \mathrm{T}+t®_{+})$ as $\bar{F}(x) = \mathrm{P}[\mathrm{X} > \mathrm{x}] = ®\ \mathrm{e}^{Tx}\, e$, $\cdot\ (u) = ®_{+} e^{(T+t®_{+})u} e$,where $®_{+} = ® \int_{0}^{\infty} (T + t®_{+})^{t} \sum_{i=1}^{3} a_i e^{-R_i t} dt$.

We also define the following,

$F(u, y) = P[T < \infty, |R(T)| \le y]$ and $\cdot\ (u, y) = P[T < \infty, |R(T)| > y]$, $F(u, \mathrm{y})$ is the probability that ruin occurs with initial surplus u and a deficit at ruin is not greater than y, see Gerber *et al.* (1987), $\cdot\ (u, y)$ is the probability that ruin occurs and deficit at ruin exceeds y. Recalling necessary information;

$$\cdot\ (u, y) = P[M > u, |R(t)| > y]$$

$$= P[M > u].\mathrm{P}[|R(T)| > y]$$

$$= ®_{+} e^{(T+t®_{+})u} e T^{y} e$$, using the definition of p.g.f

Now $F_u(y)$ -distribution of the deficit at ruin,ie.

$$F_u(y) = P[R(t) \le y / R(t) < 0]$$

$$= \frac{F(u, y)}{\cdot\ (u)}$$

$$= \frac{\cdot\ (u) - \cdot\ (u, y)}{\cdot\ (u)}$$

$$= 1 - \frac{\cdot\ (u, y)}{\cdot\ (u)}$$

$$\therefore \bar{F}_u(y) = \frac{\cdot\ (u, y)}{\cdot\ (u)}$$

Or $$\bar{F}_u(y) = \frac{®_{+}.e^{(T+t®_{+})\mathrm{u}}.e.T^{u}}{\cdot\ (u)}$$

Summary and Conclusion

In this study we emphasis the Phase type distribution and its application in risk renewal theory. The Lindley distribution is used for approximating claim size distribution. Various kinds of phase type distribution are discussed and its respective distribution forms are also derived. Two sided risk renewal theory is presented and thereafter the ruin probability form under phase type is derived. Deficit at ruin distribution and joint distribution of ruin occurring and deficit at ruin is also derived.

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