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RESEARCH ARTICLE

NONPARAMETRIC APPROACH TO ESTIMATION IN LINEAR REGRESSION

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ABSTRACT

Some estimators are suggested for slope coefficient in simple linear regression model using nonparametric approach. The estimators are based on various kinds of distances among ordered predictor variables. The mean and variance of the suggested estimators are obtained. Their efficiencies are established by carrying out comparisons among them and with their competitors including least square estimate given in the literature. The estimators and their relative efficiencies are computed for datasets for illustration.

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INTRODUCTION

Suppose (x_i, y_i), $i = 1, 2, \cdots, n$ are taken from bivariate distribution $F(x, y)$, the simplest relational form among x and y is given by simple linear regression model,

$$y_i = \alpha + \beta x_i + e_i \tag{1}$$

where, y_i is response variable, x_i is predictor variable, α is intercept parameter, β is slope parameter, e_i is independent and identically distributed random errors with zero mean and finite variance σ^2. The slope parameter β represents rate of change in y with respect to x. The simple linear regression deals with examining the linear dependency of response variable on predictor variable. The situations like investigating per capita income on consecutive years, the yield of a crop on temperature, son's height on father's height, expenditure of a household on its income, etc. constitute simple linear regression models. The aim of the present work is to suggest some estimators to β from nonparametric perspective, so that, β is estimated in a more general scenario.

Legendre (1805) and Gauss (1809) developed a popular method known as method of least squares for estimating regression coefficients in simple linear regression model by minimizing the error sum of squares, $\sum_{i=1}^{n} e_i^2$. The least square estimator is given by

$$\hat{\beta}_{ul} = \frac{\sum_{i=1}^{n}(dl_i)(y_i - \bar{y})}{\sum_{i=1}^{n}(dl_i)^2} \tag{2}$$

where, $dl_i = (x_i - \bar{x})$, $\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$ and $\bar{y} = \frac{\sum_{i=1}^{n} y_i}{n}$. Harter (1974) and Hald (1998) observed that, Euler (1749) and Mayer (1750) have developed the method of dividing the observational equations into as many groups as there are parameters to be estimated for obtaining estimators for regression coefficients in linear regression model. The division is carried out by arranging predictor variable in ascending order. Laplace (1787) generalized this method to estimate four unknown parameters in the model.

Bose (1938) developed estimation procedures for estimating slope parameter in simple linear regression model. For the methods proposed, he assumed that the x observations are at equal distances, d. For $n = 2m$, that is, when observations are even, the estimator using method of successive difference is given by

$$\hat{\beta}_{es} = \frac{\sum_{i=1}^{m}(y_{2i}-y_{2i-1})}{md} \tag{3}$$

the method of differences at half range is given by

$$\hat{\beta}_{eh} = \frac{\sum_{i=1}^{m}(y_{m+i}-y_i)}{m^2 d} \tag{4}$$

and method of range is given by

$$\hat{\beta}_{er} = \frac{y_n - y_1}{(n-1)d} \tag{5}$$

When observations are odd, i.e. $n = 2m+1$, last observation is excluded in the method of successive difference. The middle observation is eliminated in the method of differences at half range, numerator of $\hat{\beta}_{eh}$ is replaced by $\sum_{i=2}^{m+1}(y_{m+i} - y_{i-1})$ and denominator by $m(m+1)d$ in (4). These estimators are compared with least square estimate $\hat{\beta}_{el} = \frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\frac{n(n^2-1)}{12}}$ when x_i's are taken at unit distance. However, if equal distance d is taken into consideration, $\hat{\beta}_{el}$ is given by $\hat{\beta}_{el} = \frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{d^2\frac{n(n^2-1)}{12}}$. The relative efficiencies of each of the proposed estimator with the other are obtained. Nair and Shrivastava (1942) generalized the procedure given by Bose (1938) by dividing error equations in optimum number of groups to increase the relative efficiency of the estimates.

In this paper, we suggest some estimators for β in section 2. The mean and variance of these estimators are given in section 3 and in section 4 the performance of the estimators using their relative efficiencies is investigated. Section 5 deals with illustration and section 6 contains conclusions.

SUGGESTED ESTIMATORS

The methods proposed by Bose (1938) are applicable only for the data sets where x_i's are at equal distances, such as finding the inflation or gross domestic product (GDP) of a country in successive years, observing stocks and shares in successive quarters, noticing the growth of certain microbes at equal differences of increasing temperature, etc. However, in numerous situations, the distances among x_i's need not be equal. Bose's estimators can be modified to include such situations. Hence, we suggest some estimators including adaptive Bose's estimators in nonparametric viewpoint.

Given a bivariate data set, suppose x_i's $i = 1,2,\cdots,n$ are arranged in increasing order, then i^{th} order statistic is represented by $x_{(i)}$. Let the corresponding y observation be y_i^*. Then the adaptive Bose's estimators are $\hat{\beta}_{us}$, $\hat{\beta}_{uh}$ and $\hat{\beta}_{ur}$ due to method of successive differences, method of differences at half range and method of range respectively, when distances among x_i's are unequal. The following estimators are suggested when $n = 2m$.

$$\hat{\beta}_{us} = \frac{\sum_{i=1}^{m}(y_{2i}^*-y_{2i-1}^*)}{\sum_{i=1}^{m} ds_i}, \quad ds_i = x_{(2i)} - x_{(2i-1)} \tag{6}$$

$$\hat{\beta}_{uh} = \frac{\sum_{i=1}^{m}(y_{m+i}^*-y_i^*)}{\sum_{i=1}^{m} dh_i}, \quad dh_i = x_{(m+i)} - x_{(i)} \tag{7}$$

and

$$\hat{\beta}_{ur} = \frac{(y_{2m}^*-y_1^*)}{(x_{(2m)}-x_{(1)})} \tag{8}$$

The above estimators reduce to Bose's estimators when distances among $x_{(i)}$'s are equal. Further, some other estimators based on different kinds of distances among $x_{(i)}$'s is suggested. The estimator taking difference among consecutive $x_{(i)}$'s is given by

$$\hat{\beta}_{uc} = \frac{\sum_{i=1}^{2m-1}(y_{i+1}^*-y_i^*)}{\sum_{i=1}^{2m-1} dc_i}, \quad dc_i = \left(x_{(i+1)} - x_{(i)}\right) \tag{9}$$

the estimator based on taking differences among ordered observations lying equally on either side of the half range is given by

$$\hat{\beta}_{ud} = \frac{\sum_{i=1}^{m}(y_{m+i}^*-y_{m-(i-1)}^*)}{\sum_{i=1}^{m} dd_i}, \quad dd_i = x_{(m+i)} - x_{(m-(i-1))} \tag{10}$$

and the estimator based on difference among odd ordered observations and even ordered observations is given by

$$\hat{\beta}_{ue} = \frac{\sum_{i=3}^{n}(y_i^*-y_{i-2}^*)}{\sum_{i=3}^{n} de_i}, \quad de_i = x_{(i)} - x_{(i-2)}\ . \tag{11}$$

Similarly, for $= 2m+1$, the last pair of observations is omitted in the method of successive difference, the middle pair of observation is excluded in the method of half range and $\hat{\beta}_{uh}$ is replaced by $\sum_{i=2}^{m+1}(y_{m+i}^* - y_{i-1}^*)$ in the numerator and by $\sum_{i=2}^{m+1}(x_{(m+i)} - x_{(i-1)})$ in the denominator of (7). Also, $\hat{\beta}_{ur}$ is obtained by replacing $2m$ by $2m+1$ in (8), replacing the summation by $\sum_{i=1}^{2m}$ in (9) and replacing $\sum_{i=2}^{m+1}(y^*_{m+i} - y^*_{m-(i-2)})$ in the numerator and $\sum_{i=2}^{m+1}(x_{(m+i)} - x_{(m-(i-2))})$ in the denominator of (10). We observe that, $\hat{\beta}_{ue}$ remains unchanged when *n* is odd, $\hat{\beta}_{uc}$ reduces to $\hat{\beta}_{ur}$ and $\hat{\beta}_{ud}$ reduces to $\hat{\beta}_{uh}$. Also, when distances are equal, $\hat{\beta}_{ue}$ is reduced to $\hat{\beta}_{ee}$ given by

$$\hat{\beta}_{ee} = \frac{\sum_{i=3}^{n}(y^*_i - y^*_{i-2})}{2(n-2)d} \tag{12}$$

MEAN AND VARIANCE OF THE SUGGESTED ESTIMATORS

In this section, we discuss about the unbiasedness of the suggested estimators, viz. $\hat{\beta}_{us}$, $\hat{\beta}_{uh}$, $\hat{\beta}_{ur}$, $\hat{\beta}_{uc}$, $\hat{\beta}_{ud}$, $\hat{\beta}_{ue}$, $\hat{\beta}_{ee}$ and obtain their variances. The mean of $\hat{\beta}_{us}$ is given by

$$E(\hat{\beta}_{us}) = E\left(\frac{\sum_{i=1}^{m}(y^*_{2i} - y^*_{2i-1})}{\sum_{i=1}^{m} ds_i}\right)$$

$$= \frac{1}{\sum_{i=1}^{m} ds_i} E\left(\sum_{i=1}^{m}(y^*_{2i} - y^*_{2i-1})\right)$$

$$= \frac{1}{\sum_{i=1}^{m} ds_i} \sum_{i=1}^{m} E(y^*_{2i} - y^*_{2i-1})$$

$$= \frac{\beta}{\sum_{i=1}^{m} ds_i}\left(\sum_{i=1}^{m}(x_{(2i)} - x_{(2i-1)})\right)$$

$$= \frac{\beta}{\sum_{i=1}^{m} ds_i}\left(\sum_{i=1}^{m} ds_i\right)$$

$$= \beta\ . \tag{13}$$

That is, $\hat{\beta}_{us}$ is unbiased estimator of β.

The variance of $\hat{\beta}_{us}$ is given by

$$V(\hat{\beta}_{us}) = V\left(\frac{\sum_{i=1}^{m}(y^*_{2i} - y^*_{2i-1})}{\sum_{i=1}^{m} ds_i}\right)$$

$$= \frac{1}{\left(\sum_{i=1}^{m} ds_i\right)^2} V\left(\sum_{i=1}^{m}(y^*_{2i} - y^*_{2i-1})\right) \tag{14}$$

$$= \frac{n\sigma^2}{\left(\sum_{i=1}^{m} ds_i\right)^2}\ .$$

On similar lines, we see that $E(\hat{\beta}_{uh}) = E(\hat{\beta}_{ur}) = E(\hat{\beta}_{ue}) = E(\hat{\beta}_{ee}) = \beta$ establishing that all the suggested estimators are unbiased estimator of β. However, the variances of some of these estimators differ and the variances are furnished in Table 1 below.

Table 1. Variances of suggested estimators

Estimator	Variance
$\hat{\beta}_{us}$	$\frac{n\sigma^2}{(\sum_{i=1}^{m} ds_i)^2}$
$\hat{\beta}_{uh}$	$\frac{n\sigma^2}{(\sum_{i=1}^{m} dh_i)^2}$
$\hat{\beta}_{ur}$	$\frac{2\sigma^2}{\left(x_{(2m)} - x_{(1)}\right)^2}$
$\hat{\beta}_{ue}$	$\frac{4\sigma^2}{(\sum_{i=3}^{n} de_i)^2}$
$\hat{\beta}_{ee}$	$\frac{\sigma^2}{(n-2)^2 d^2}$

From the above table, we observe that the variances are inversely proportional to the square of the sum of the distances among ordered predictor variables.

RELATIVE EFFICIENCY

In this section, we furnish the relative efficiencies of the suggested estimators with Bose's estimators and least square estimate in Table 2 and the relative efficiencies among suggested estimators in Table 3. The relative efficiency establishes the relative performance of estimators in terms of their precision. It is the ratio of precision of two estimators. i.e. suppose A and B are two unbiased estimators of β, the relative efficiency of A with respect to B is given by

$$RE(A,B) = \frac{Var(B)}{Var(A)} \tag{15}$$

If $RE(A,B) > 1$, then we interpret that A is better than B interms of its performance.

Table 2 shows that the relative efficiency of estimators given in row with respect to those given in column will be greater than unity if in each cell the expression in numerator with unequal distances is greater than the expression in denominator with distance d. Table 3 too gives similar results.

Table 2. Relative efficiencies of suggested estimators with respect to various estimators

RE wrt → of ↓			Estimators with equal distances among ordered predictor variable				
			Estimators due to Bose (1938)			Least square estimate	Suggested estimator
			$\hat{\beta}_{es}$	$\hat{\beta}_{eh}$	$\hat{\beta}_{er}$	$\hat{\beta}_{el}$	$\hat{\beta}_{ee}$
Estimators with unequal distances among ordered predictor variable	Suggested estimators	$\hat{\beta}_{us}$	$\frac{4(\sum_{i=1}^{m} ds_i)^2}{n^2d^2}$	$\frac{16(\sum_{i=1}^{m} ds_i)^2}{n^4d^2}$	$\frac{2(\sum_{i=1}^{m} ds_i)^2}{n(n-1)^2d^2}$	$\frac{12(\sum_{i=1}^{m} ds_i)^2}{(n^2-1)n^2d^2}$	$\frac{(\sum_{i=1}^{m} ds_i)^2}{n(n-2)^2d^2}$
		$\hat{\beta}_{uh}$	$\frac{4(\sum_{i=1}^{m} dh_i)^2}{n^2d^2}$	$\frac{16(\sum_{i=1}^{m} dh_i)^2}{n^4d^2}$	$\frac{2(\sum_{i=1}^{m} dh_i)^2}{n(n-1)^2d^2}$	$\frac{12(\sum_{i=1}^{m} dh_i)^2}{(n^2-1)n^2d^2}$	$\frac{(\sum_{i=1}^{m} dh_i)^2}{n(n-2)^2d^2}$
		$\hat{\beta}_{ur}$	$\frac{2(x_{(2m)}-x_{(1)})^2}{nd^2}$	$\frac{8(x_{(2m)}-x_{(1)})^2}{n^3d^2}$	$\frac{(x_{(2m)}-x_{(1)})^2}{(n-1)^2d^2}$	$\frac{6(x_{(2m)}-x_{(1)})^2}{n(n^2-1)d^2}$	$\frac{(x_{(2m)}-x_{(1)})^2}{2(n-2)^2d^2}$
		$\hat{\beta}_{ue}$	$\frac{(\sum_{i=3}^{n} de_i)^2}{nd^2}$	$\frac{4(\sum_{i=3}^{n} de_i)^2}{n^3d^2}$	$\frac{(\sum_{i=3}^{n} de_i)^2}{2(n-1)^2d^2}$	$\frac{3(\sum_{i=3}^{n} de_i)^2}{n(n^2-1)d^2}$	$\frac{(\sum_{i=3}^{n} de_i)^2}{4(n-2)^2d^2}$
	Least square estimate	$\hat{\beta}_{ul}$	$\frac{\sum_{i=1}^{n}(dl_i)^2}{nd^2}$	$\frac{16\sum_{i=1}^{n}(dl_i)^2}{n^3d^2}$	$\frac{2\sum_{i=1}^{n}(dl_i)^2}{(n-1)^2d^2}$	$\frac{12\sum_{i=1}^{n}(dl_i)^2}{n(n^2-1)d^2}$	$\frac{\sum_{i=1}^{n}(dl_i)^2}{(n-2)^2d^2}$

Table 3. Relative efficiencies among the suggested estimators

RE	Expressions	RE	Expressions
$(\hat{\beta}_{us}, \hat{\beta}_{uh})$	$\frac{(\sum_{i=1}^{m} ds_i)^2}{(\sum_{i=1}^{m} dh_i)^2}$	$(\hat{\beta}_{uh}, \hat{\beta}_{ue})$	$\frac{4(\sum_{i=1}^{m} dh_i)^2}{n(\sum_{i=3}^{n} de_i)^2}$
$(\hat{\beta}_{us}, \hat{\beta}_{ur})$	$\frac{2(\sum_{i=1}^{m} ds_i)^2}{n(x_{(2m)}-x_{(1)})^2}$	$(\hat{\beta}_{uh}, \hat{\beta}_{ee})$	$\frac{(\sum_{i=1}^{m} dh_i)^2}{n(n-2)^2d^2}$
$(\hat{\beta}_{us}, \hat{\beta}_{ue})$	$\frac{4(\sum_{i=1}^{m} ds_i)^2}{n(\sum_{i=3}^{n} de_i)^2}$	$(\hat{\beta}_{ur}, \hat{\beta}_{ue})$	$\frac{2(x_{(2m)}-x_{(1)})^2}{(\sum_{i=3}^{n} de_i)^2}$
$(\hat{\beta}_{us}, \hat{\beta}_{ee})$	$\frac{(\sum_{i=1}^{m} ds_i)^2}{n(n-2)^2d^2}$	$(\hat{\beta}_{ur}, \hat{\beta}_{ee})$	$\frac{(x_{(2m)}-x_{(1)})^2}{2(n-2)^2d^2}$
$(\hat{\beta}_{uh}, \hat{\beta}_{ur})$	$\frac{2(\sum_{i=1}^{m} dh_i)^2}{n(x_{(2m)}-x_{(1)})^2}$	$(\hat{\beta}_{ue}, \hat{\beta}_{ee})$	$\frac{(\sum_{i=3}^{n} de_i)^2}{4(n-2)^2d^2}$

ILLUSTRATION

In this section, we illustrate the suggested estimators through examples. Example 1 is due to Walters et. al.(2006), example 2 is due to Graybill (1961) and example 3 is data taken from nseindia.com and bseindia.com. For the examples under consideration, values of $\hat{\beta}$ and relative efficiencies of the estimators are obtained using R-programming. To fit the regression model given in (1), the intercept parameter α is estimated by $\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x}$. Using various suggested estimators and least square estimate of β, the regression lines are plotted for example 1, 2 and 3 respectively in figures 1, 2 and 3.

Example 1. Dataset from Walters et. al. (2006). In this dataset, the variable x is number of boat registration and y is number of manatees killed by boat in Florida from 1977 to 1990.

x	447	460	481	498	512	513	526	559	585	614	645	675	711	719
y	13	21	24	16	20	24	15	34	33	33	39	43	50	47

Using equation (2), (6), (7), (8) and (11), $\hat{\beta}_{ul}$, $\hat{\beta}_{us}$, $\hat{\beta}_{uh}$, $\hat{\beta}_{ur}$ and $\hat{\beta}_{ue}$ are obtained respectively and are presented in Table 4. Also, the relative efficiencies among suggested estimators and with least square estimate are given in Table 4.

Table 4. Value of estimators and their RE

Estimator	Value	RE	Value	RE	Value
$\hat{\beta}_{us}$	0.1832	$(\hat{\beta}_{us}, \hat{\beta}_{uh})$	0.0150	$(\hat{\beta}_{uh}, \hat{\beta}_{ue})$	1.1981
$\hat{\beta}_{uh}$	0.1363	$(\hat{\beta}_{us}, \hat{\beta}_{ur})$	0.0331	$(\hat{\beta}_{uh}, \hat{\beta}_{ul})$	0.7461
$\hat{\beta}_{ur}$	0.1250	$(\hat{\beta}_{us}, \hat{\beta}_{ue})$	0.0179	$(\hat{\beta}_{ur}, \hat{\beta}_{ue})$	0.5410
$\hat{\beta}_{ue}$	0.1204	$(\hat{\beta}_{us}, \hat{\beta}_{ul})$	0.0112	$(\hat{\beta}_{ur}, \hat{\beta}_{ul})$	0.3369
$\hat{\beta}_{ul}$	0.1249	$(\hat{\beta}_{uh}, \hat{\beta}_{ur})$	2.2148	$(\hat{\beta}_{ue}, \hat{\beta}_{ul})$	0.6227

It is observed that $E(\hat{\beta}_{us}, \hat{\beta}_{uh})$ = 0.0150 < 1, which means $\hat{\beta}_{us}$ is under performing when compared with $\hat{\beta}_{uh}$. That is $\hat{\beta}_{uh}$ is 98.5% better than $\hat{\beta}_{us}$. The $RE(\hat{\beta}_{uh}, \hat{\beta}_{ur}) = 2.2148 > 1$, which means $\hat{\beta}_{uh}$ is 121.48% better than $\hat{\beta}_{ur}$. All other RE values can be interpreted on similar grounds. It is observed that $\hat{\beta}_{uh}$ is better than $\hat{\beta}_{us}$, $\hat{\beta}_{ur}$, $\hat{\beta}_{ue}$; $\hat{\beta}_{ue}$ is better than $\hat{\beta}_{us}$, $\hat{\beta}_{ur}$ and $\hat{\beta}_{ur}$ is better than $\hat{\beta}_{us}$. From Figure 1, it is observed that, among the regression lines obtained using suggested estimators, the regression line based on $\hat{\beta}_{uh}$ is a better fit to the given data.

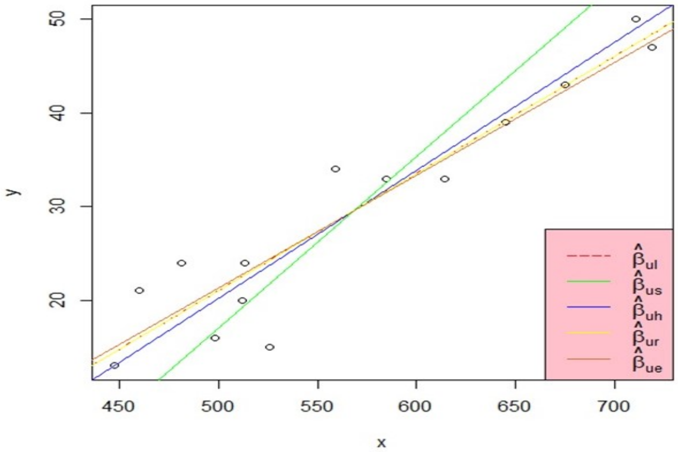

Figure 1. Fitting of regression lines based on various estimators for data given in example 1

Example 2. Dataset from Graybill (1961). This data is explaining the distance of a particle y, travelling with time x.

x	1	2	3	4	10	12	18
y	9	15	19	20	45	55	78

The computed values of estimators and the relative efficiencies of various estimators based on above data are given in Table 5.

Table 5. Value of estimators and their RE

Estimator	Value	RE	Value	RE	Value
$\hat{\beta}_{us}$	4.25	$(\hat{\beta}_{us}, \hat{\beta}_{uh})$	0.0138	$(\hat{\beta}_{uh}, \hat{\beta}_{ue})$	1.0572
$\hat{\beta}_{uh}$	3.97	$(\hat{\beta}_{us}, \hat{\beta}_{ur})$	0.0185	$(\hat{\beta}_{uh}, \hat{\beta}_{ul})$	0.7999
$\hat{\beta}_{ur}$	4.05	$(\hat{\beta}_{us}, \hat{\beta}_{ue})$	0.0146	$(\hat{\beta}_{ur}, \hat{\beta}_{ue})$	0.7929
$\hat{\beta}_{ue}$	4.03	$(\hat{\beta}_{us}, \hat{\beta}_{ul})$	0.0111	$(\hat{\beta}_{ur}, \hat{\beta}_{ul})$	0.5999
$\hat{\beta}_{ul}$	4.02	$(\hat{\beta}_{uh}, \hat{\beta}_{ur})$	1.3333	$(\hat{\beta}_{ue}, \hat{\beta}_{ul})$	0.7567

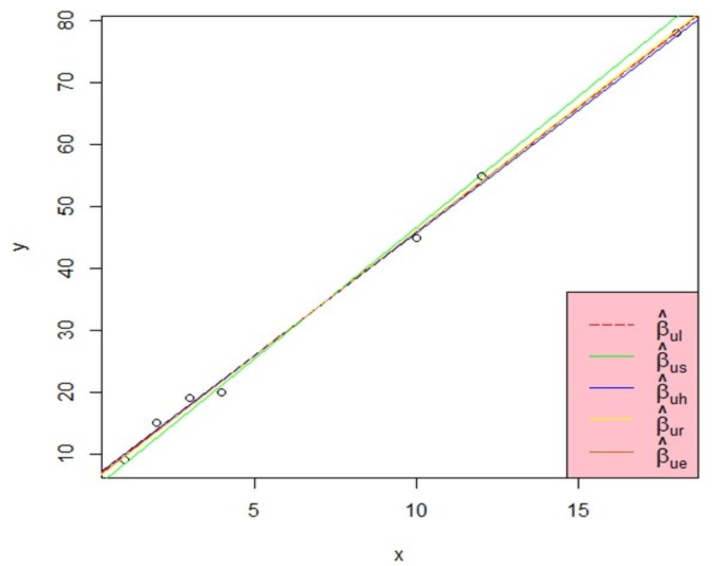

Figure 2. Fitting of regression lines based on various estimators for data given in example 2

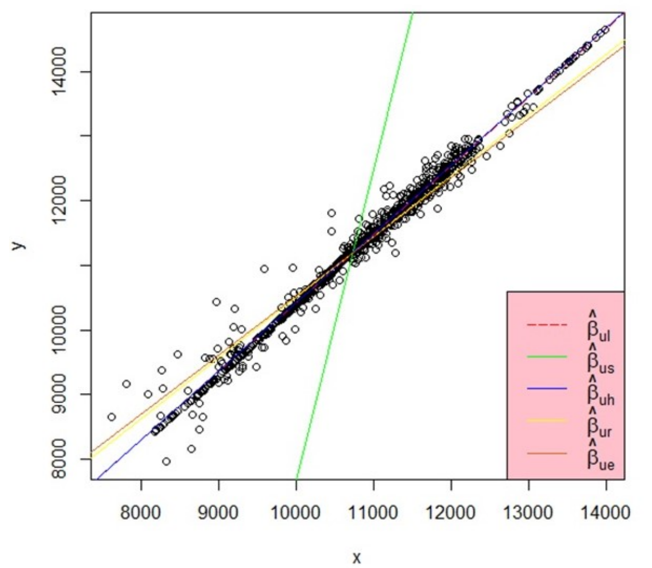

Figure 3. Fitting of regression lines based on various estimators for data given in example 3

Table 5 reveals that, $\hat{\beta}_{uh}$ is better than $\hat{\beta}_{us}$, $\hat{\beta}_{ur}$, $\hat{\beta}_{ue}$; $\hat{\beta}_{ue}$ is better than $\hat{\beta}_{us}$, $\hat{\beta}_{ur}$ and $\hat{\beta}_{ur}$ is better than $\hat{\beta}_{us}$. The same results are observed in example 1 too. From the figure 2, it is observed that, the regression lines obtained using $\hat{\beta}_{uh}$, $\hat{\beta}_{ur}$ and $\hat{\beta}_{ul}$ are identical and give better fit to given data.

Example 3. Dataset from website nseindia.com and bseindia.com. This data is concerned with daily closing prices of index NIFTY50 from National stock exchange (NSE) and SENSEX50 from Bombay stock exchange (BSE). The data consist 989 observations of 4 years from 2017 to 2020. The variable x is daily closing price of index NIFTY50 from NSE and the variable y is daily closing price of index SENSEX50 from BSE. Since the data is large, we furnish the fitting of regression lines based on various estimates. The values of the estimates are given by $\hat{\beta}_{us}$= 4.8177, $\hat{\beta}_{uh}$= 1.0624, $\hat{\beta}_{ur}$= 0.9392, $\hat{\beta}_{ue}$= 0.9133 and $\hat{\beta}_{ul}$= 1.0580.

From Figure 3, it is observed that, the regression line obtained using $\hat{\beta}_{uh}$ and $\hat{\beta}_{ul}$ coincide and are better fit to the data. Also, value of $\hat{\beta}_{uh}$ is near to the value of $\hat{\beta}_{ul}$.

CONCLUSION

In this section, we record our conclusions based on the results of our findings.

- The suggested estimators are based on various types of distances among predictor variables.
- These estimators are the functions of order statistics when predictor variable have unequal distances.
- These estimators are applicable in many realistic situations and are generalization to Bose's (1938) estimators.
- The estimators $\hat{\beta}_{us}$, $\hat{\beta}_{uh}$, $\hat{\beta}_{ur}$, $\hat{\beta}_{uc}$, $\hat{\beta}_{ud}$, $\hat{\beta}_{ue}$ and $\hat{\beta}_{ee}$ which are based on different kinds of distances reveal that, $\hat{\beta}_{uc}$ reduces to $\hat{\beta}_{ur}$ and $\hat{\beta}_{ud}$ reduces to $\hat{\beta}_{uh}$.
- $\hat{\beta}_{us}$, $\hat{\beta}_{uh}$ and $\hat{\beta}_{ur}$ reduces to Bose's (1938) estimators $\hat{\beta}_{es}$, $\hat{\beta}_{eh}$ and $\hat{\beta}_{er}$ respectively when distances among ordered predictor variables are equal.
- All the suggested estimators are unbiased estimators of β.
- Among the suggested estimators, $\hat{\beta}_{uh}$ outperforms $\hat{\beta}_{us}$, $\hat{\beta}_{ur}$, and $\hat{\beta}_{ue}$. Also, $\hat{\beta}_{ue}$ is better than $\hat{\beta}_{us}$, $\hat{\beta}_{ur}$ and $\hat{\beta}_{ur}$ is better than $\hat{\beta}_{us}$.
- Among the suggested estimators, the regression line based on $\hat{\beta}_{uh}$ is a better fit to the data.
- For large data, the regression lines fitted using $\hat{\beta}_{uh}$ and $\hat{\beta}_{ul}$ coincide.

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