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RESEARCH ARTICLE

NUMERICAL SIMULATION OF SAINT VENANT-EXNER EQUATION WITH MOVING MESH METHOD

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ABSTRACT

The moving mesh method is used to simulate the Saint Venant-Exner equation. Like the exact solution is unavailable and in order to illustrate the efficiency and accuracy property of this method, we compare the numerical solution with a reference one obtained using a very fine mesh. The results obtained give good agreement between reference solutions and numerical solutions.

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INTRODUCTION

Water needs require the recovery of surface water, which is why the construction of drainage channels, bridges and dams is very important.The Saint-Venant–Exner system is used to model sediment transport phenomenon that occurs in large time and space scales in river hydraulics or coastal studies. It's therefore necessary to study the interaction between flow and transport dynamics and how this relationship influences surface morphological changes (1). The study of sediment transport focuses on understanding the relationship that exists between the movement of water and the movement of sedimentary materials.It is therefore essential to develop an approach capable of preventing the difficulties that this process may cause. In the numerical simulation of Saint Venant-Exner problem, the mathematical model includes a hydrodynamical component coupled with a morphodynamical component (2, 4).The Saint-Venantequations are used to predict the hydrodynamic behavior of water flows while the Exner equation is used to model sediment transport and simulate bed morphology changes (3, 5). The numerical simulation involves different physical mechanisms, hence, have robust numerical schemes is need to accurately resolve both hydrodynamic and morphodynamic problem in time and space. The morphodynamical flows involve important features such as moving fronts, stiff fronts, shock waves which present a significant challenge to the numerical accuracy and stability. The aim of this work is to develop a simple, but robust, stable, reliable and accurate numerical method able that can accurately handle the discontinuities and capture shocks. The outline of the paper is as follows.

In Section 2we give some details on the governing Saint Venant-Exner equation.

In **Section 3** we make a recall of the moving mesh method and numerical schemes.
In **Section 4**, we present a serial of numerical test problems to evaluate both efficiency, accuracy and performance of the method.We present some conclusions in Section 5.

Governing Saint Venant-Exner equation: The Saint Venant- Exner equation is obtained from the coupling of hydrodynamic model and morphodynamical component (6, 7). The hydrodynamic model is described by the Saint-Venant equation and the morphodynamic model is described by the Exner equation which includes a conservation law related to the evolution of the bottom topography due to the fluid action (8, 9).

The governing equations are obtained under Saint-Venant system conditions and includes equations for the conservation of water mass and momentum of the water phase.For an inviscid and incompressible flow, the hydrodynamic modelized by the Saint-Venant system is as follow (10):

$$\begin{cases} \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0, & (1) \\ \frac{\partial}{\partial t}(hu) + \frac{\partial}{\partial x}\left(hu^2 + \frac{1}{2}gh^2\right) = gh(S-\tau), & (2) \end{cases}$$

where τ is bottom frictional terms. The sediment dynamics can be expressed by a simple sediment continuity or Exner equation.Mathematically, this equation is given as:

$$\frac{\partial B}{\partial t} + \xi \frac{\partial Q_s}{\partial x} = 0 \qquad (3)$$

For a fixed bed, by neglecting viscous and Coriolis effects, the one-dimensional PDE system of Saint Venant-Exner model is:

$$\begin{cases} \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0 \\ \frac{\partial}{\partial t}(hu) + \frac{\partial}{\partial x}\left(hu^2 + \frac{1}{2}gh^2\right) = ghS\,, \\ \frac{\partial B}{\partial t} + \rho \frac{\partial Q_s}{\partial x} = 0 \end{cases} \qquad (4)$$

where τ is bottom frictional terms.

where h is the water depth, u is the velocity component, g is the acceleration due to gravity, B represent the thickness of sediment layer, $S = -\frac{\partial B}{\partial x}$ is the bed slope, discharge Q_s is the volumetric bedload sediment transport rate per unit time and width $\rho = \frac{1}{1-\gamma}$, where γ is the porosity of the sediment layer, the conservative variable $q = hu$ is called water discharge (10, 11,12).Grass has proposed the formula which predicted and estimated bedload transport rate (13). Expression proposed consider Q_s as a function of the flow velocity and a coefficient which depends on soil properties.

The expression is $Q_s = A_g u|u|^{m_g - 1}$ where $1 \leq m_g \leq 4$ and $0 \leq A_g \leq 1$. The parameter A_g is the coefficient which control the intensity of interaction between flow and sediment particles (14, 15).

The moving mesh methodand numerical schemes

Brief recall of moving mesh method: Numerical techniques to solve time partial differential equation are most often based on a discretization of the spatial domain and the resulting mesh is generally fixed in time. Sometime the mesh itself need to change as the system evolves, adapting to the physics problem. The adaptive method is more efficient for numerical solution of time partial differential equations that develop stiff fronts, shock waves, or overflow, that are localized in space (16, 17). Adapting the mesh can prove computationally efficient in that an adaptive mesh generally requires few points than a fixed mesh to attain the same degree of accuracy. The main idea of moving grids method is to relocate grid points in a mesh having a fixed number of nodes in such a way that the nodes remain concentrated in regions of rapid variation of the solution.The fundamental principle of moving mesh method is the equidistribution principle proposed by Carl de Boor, principle which offer an excellent error estimation principle when formulating moving mesh equations. The grid points are moved so that a specified quantity called monitor function is equally distributed over the spatial domain (18). In the moving mesh method, the monitor function is chosen to redistribute more grid points at critical regions where more accuracy is needed there by reducing errors introduced by the numerical scheme (19, 20, 21, 22). The arc-length monitor function is used in this paper.

Numerical formulation of the moving mesh method: Let's $[a;\ b]$ be the physical domain with a physical variable x and $[0;\ 1]$ be the computational domain for a computational variable ξ. The coordinates transform is expressed as follow:

$$x = x(\xi;\ t): [0,1] \longrightarrow [a,b], t > 0, x \in [a;b], \xi \in [0;1]$$

The variables h, u, B are transformed as:

$$h(x, \quad t) = h(x(\xi,t),t)\,u(x,\ t) = u(x(\xi,t),t) \qquad (5)$$
$$B(x,t) = B(x(\xi,t),t)$$

The coordinate x is rearranged as follows:

$$x_i(\xi) = x(\xi_i,t), i = 1, \ldots, n+1.$$

$$\xi_i = \frac{(i-1)(b-a)}{n}, i = 1, \dots, n+1$$.

The uniform mesh on $[0, 1]$ is ξ_i and $a = x_1 < x_2 < \cdots < x_n < x_{n+1} = b$ is the corresponding mesh on physical domain. Applying the chain rule of the method

$$h_x = \frac{h_\xi}{x_\xi}, \quad h_t = \dot{h} - \frac{h_\xi}{x_\xi} x_t$$

$$q_x = \frac{q_\xi}{x_\xi}, \quad q_t = \dot{q} - \frac{q_\xi}{x_\xi} x_t$$

$$B_x = \frac{B_\xi}{x_\xi}, \quad B_t = \dot{B} - \frac{B_\xi}{x_\xi} x_t$$

and posing that $q = hu$, $Q_s = u^3$, the Saint Venant-Exner system (4) can be written as follows:

$$\begin{cases} \dfrac{\partial h}{\partial t} - \dfrac{h_\xi}{x_\xi} x_t + \dfrac{q_\xi}{x_\xi} = 0, (6) \\ \dfrac{\partial q}{\partial t} - \dfrac{q_\xi}{x_\xi} x_t + \dfrac{2hqq_\xi - q^2 h_\xi}{h^2 x_\xi} + gh\left(\dfrac{h_\xi}{x_\xi} + \dfrac{B_\xi}{x_\xi}\right) = 0, (7) \\ \dfrac{\partial B}{\partial t} - \dfrac{B_\xi}{x_\xi} x_t + \dfrac{3A_g}{1-\gamma}\left(\dfrac{q}{h}\right)^2 \dfrac{hq_\xi - qh_\xi}{h^2 x_\xi} = 0, \quad (8) \end{cases}$$

Employed method of lines approach and using central finite difference scheme for space variable discretization, the system of ODEs obtained is as follows:

$$\begin{cases} \dfrac{dh_i}{dt} - \dfrac{h_{i+1} - h_{i-1}}{x_{i+1} - x_{i-1}} \dfrac{dx_i}{dt} + \dfrac{q_{i+1} - q_{i-1}}{x_{i+1} - x_{i-1}} = 0; \qquad i = 2, \dots, n \\ \dfrac{dq_i}{dt} - \dfrac{q_{i+1} - q_{i-1}}{x_{i+1} - x_{i-1}} \dfrac{dx_i}{dt} + \dfrac{2q_i}{h_i} \dfrac{q_{i+1} - q_{i-1}}{x_{i+1} - x_{i-1}} - \left(\dfrac{q_i}{h_i}\right)^2 \dfrac{h_{i+1} - h_{i-1}}{x_{i+1} - x_{i-1}} + gh_i\left(\dfrac{h_{i+1} - h_{i-1}}{x_{i+1} - x_{i-1}} - \dfrac{B_{i+1} - B_{i-1}}{x_{i+1} - x_{i-1}}\right) = 0, (9) \\ \dfrac{dB_i}{dt} - \dfrac{B_{i+1} - B_{i-1}}{x_{i+1} - x_{i-1}} \dfrac{dx_i}{dt} + \dfrac{3A_g}{1-\gamma}\left(\dfrac{q_i}{h_i}\right)^2 \left[\dfrac{1}{h_i} \dfrac{q_{i+1} - q_{i-1}}{x_{i+1} - x_{i-1}} - \dfrac{q_i}{h_i^2} \dfrac{h_{i+1} - h_{i-1}}{x_{i+1} - x_{i-1}}\right] = 0 \end{cases}$$

We apply MATLAB-based Method of Lines (MATMOL) toolbox specially MATLAB solver ode15s and for the study case, we give summary computational statistics using the following notations:

$\boldsymbol{n}$: moving grid node number,
$\boldsymbol{nr}$: grid fixe node number,
$\boldsymbol{STEPS}$: number of successful steps,
$\boldsymbol{FAIL}$: number of failed attempts,
$\boldsymbol{FNS}$: number of function evaluations,
$\boldsymbol{PDR}$: number of partial derivatives,
$\boldsymbol{LU}$: number of LU decompositions,
$\boldsymbol{LIN}$: number of solutions of linear system,
$\boldsymbol{CPU}$: CPU-time.

Numerical results: In this section we present numerical results obtained with some examples. Since the exact solution is unknown, we compare the computed solutions with a reference one obtained by using a very fine meshfor both h, q and B.

Example 1: Let us consider the initial condition given by:

$$h(x,0) = 1.1 + 0.1\sin(4\pi x) - 0.1\cos(2\pi x), 0 \le x \le 1$$
$$q(x,0) = 0$$
$$B(x,0) = 0.1\, cos\,(2\pi x)$$

With using periodic boundary conditions up to time t = 0.04 and constants values:

$\gamma = 0.4;\ A_g = 0.0005;\ g = 10;\ a = 0;\ b = 1.$

Figure 1 show the comparisons between the simulated results and the reference solution using 3000 points.

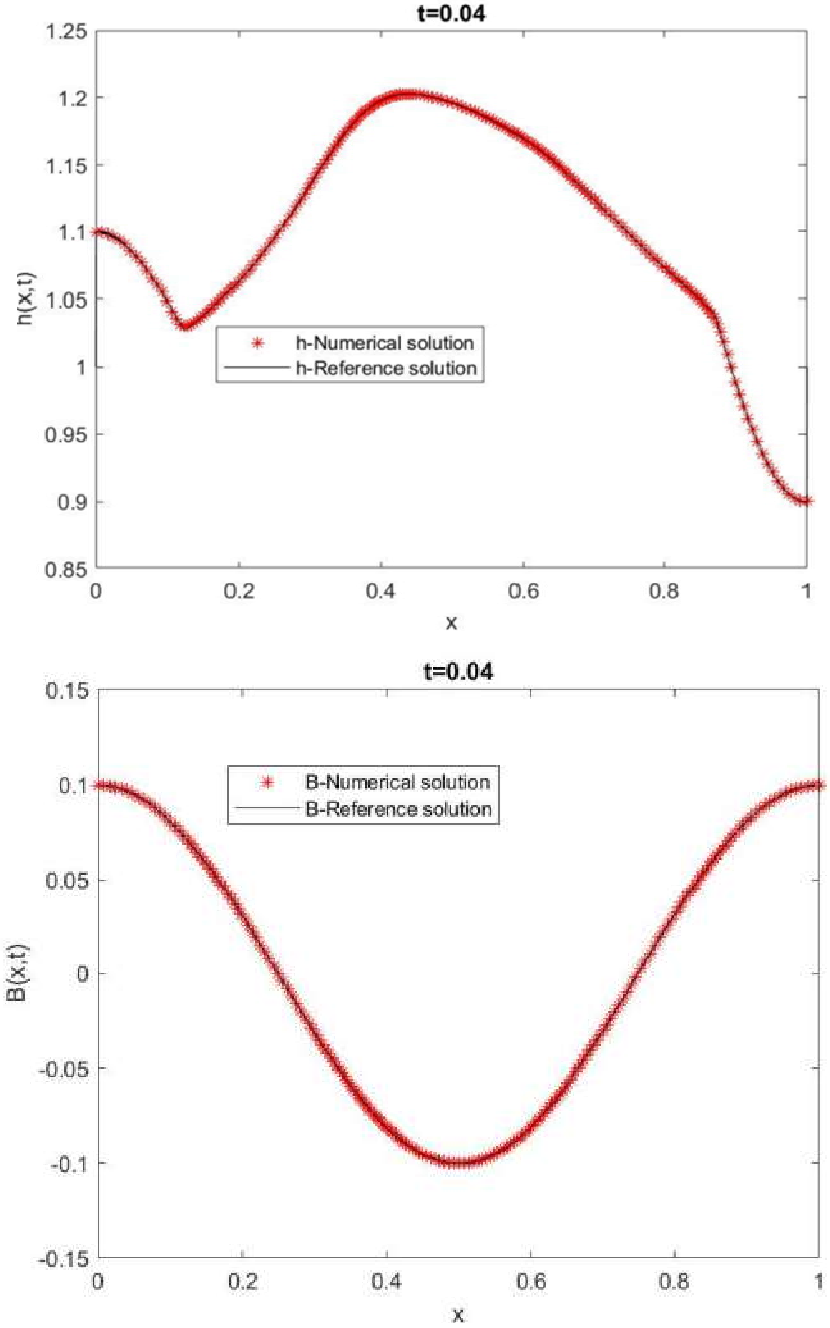


Table 1. Computational statistics of Saint Venant-Exner system

	$Suc.St$	$Fail.at$	$Fun.ev$	$Part.der$	$LU.dec$	$Sol.lin$	$CPU.t$
$n = 200$	269	96	1872	59	150	692	50.6470
$nr = 3000$	838	23	1503	1	69	1481	254.3584

From Figure 1 on can observed that the results using the proposed scheme are in very good agreement with the reference results. Table 1 shows that numerical results are very satisfactory compared with those obtained by a very large number of nodes for a fixed grid.

Example 2: We consider a numerical test consisting of a transcritical flow with a shock over a parabolic bump The initial conditions are given by (25, 26):

$$h(x,0) = \begin{cases} 0.13 + 0.05(x-10)^2, & if\ 8 < x < 12 \\ 0.33 & othewise \end{cases}$$

$$q(x,0) = 0.18$$

$$B(x,0) = \begin{cases} 3 - 0.05(x-10)^2, & if\ 8 < x < 12 \\ 2.8 & otherwise \end{cases}$$

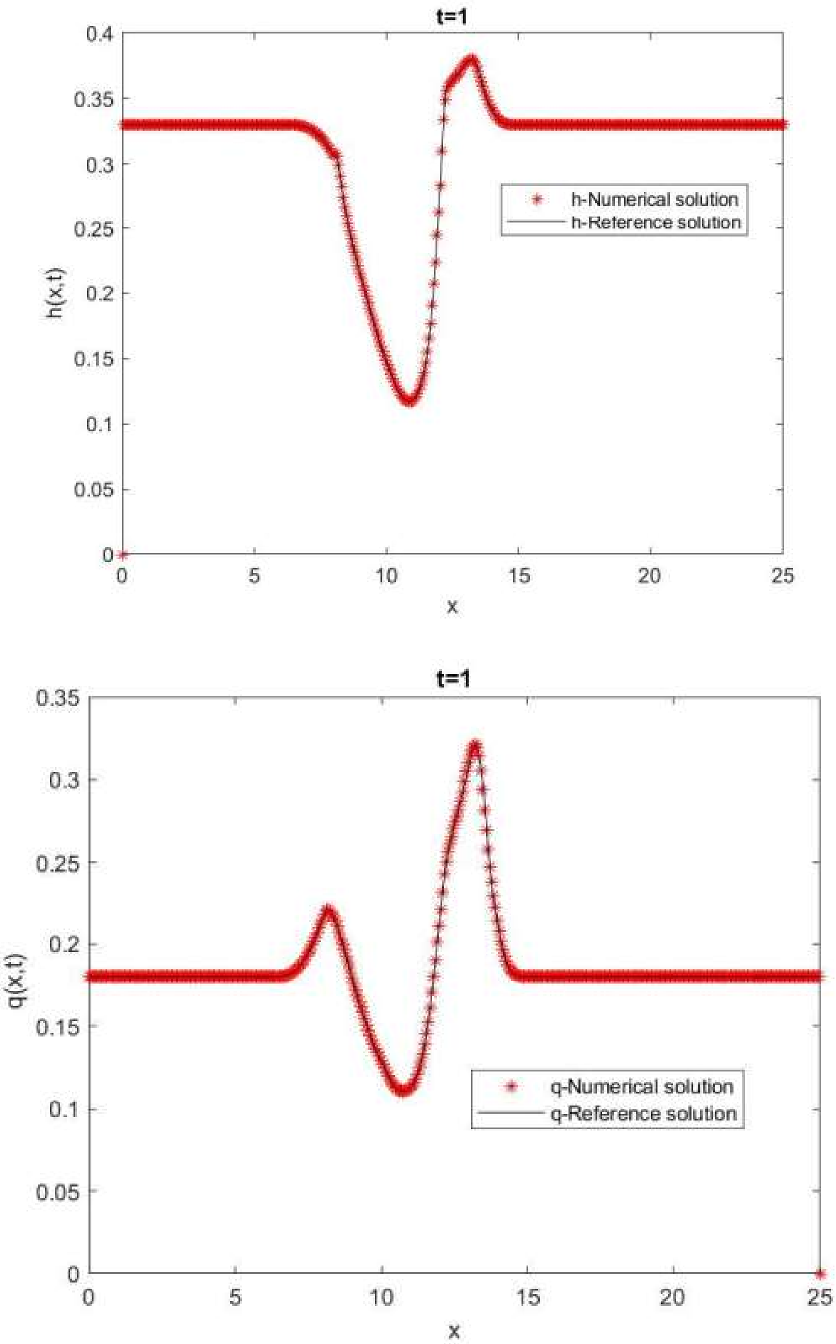
The boundary conditions are given by:

$$q(a,t) = 0.18;\ h(b,t) = 0.33$$

The constants values are:

$$\gamma = 0.004;\ A_g = 0.1;\ g = 9.812;\ a = 0;\ b = 25.$$

We compute the numerical solution using $n = 400$ points in the interval $[0, 25]$ and compare the results with the reference solution computed on a fine grid with $nr = 3000$ points attime $t = 1$. The time evolution of the bedload bottom topography, water depth and discharge are shown in Figure 2.



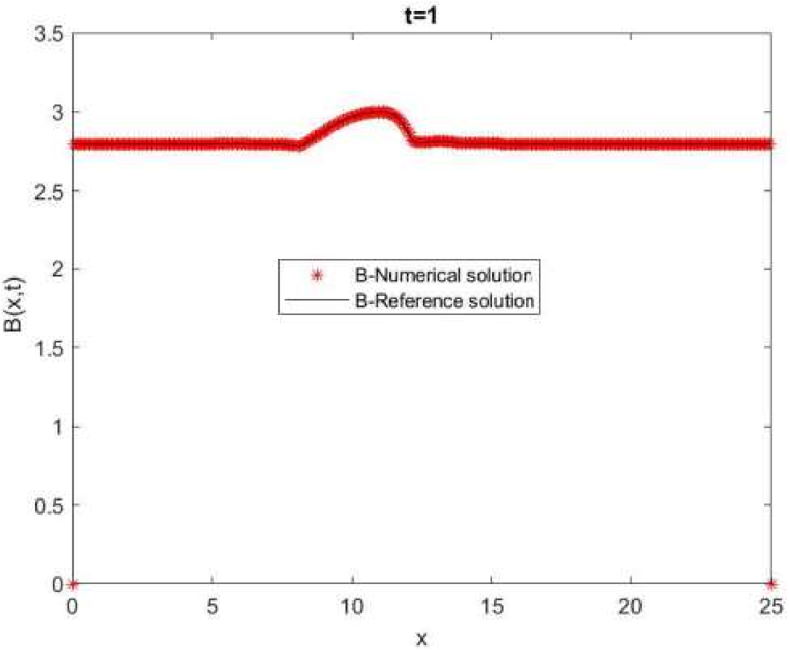


Table 3. Computational statistics for the transcritical flow with a shock case

	$Suc.St$	$Fail.at$	$Fun.ev$	$Part.der$	$LU.dec$	$Sol.lin$	$CPU.t$
$n = 400$	48	1	102	1	14	82	13.5732
$nr = 3000$	36	2	102	2	10	67	268.1138

In figure 3 we remark that when the mesh is very refined the numerical solutions converge to the reference solution. Table 3 shows that the method gives satisfactory results.

CONCLUSION

In this paper, we have discussed on moving mesh method for numerical solution of Saint Venant-Exner equation. We have presented in this paper moving mesh scheme for the simulation of Saint Venant-Exner in the one-dimensional case. Since the exact solution is unknown, the computed solutions are compared with a reference one obtained using a very fine mesh.The results of the proposed scheme based on moving mesh method in this paper shown that, numerical solutions obtained are in a good agreement with the reference solution.

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