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RESEARCH ARTICLE

A STUDY ON ADJOINT OF TYPE-2 TRIANGULAR FUZZY MATRICES

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ABSTRACT

Type-2 fuzzy sets are fuzzy sets whose membership values are fuzzy sets on the interval (0, 1). This concept was proposed by Zadeh, as an extension of fuzzy sets. Type-2 fuzzy sets possess a great expressive power and are conceptually quite appealing. Also fuzzy matrices play an important role in scientific developments. In this paper, adjoint of type-2 triangular fuzzy matrices (T2TFM) is proposed. Some more special properties of adjoint of T2TFM which are useful for finding inverse of T2TFM are presented. Numerical example is also included.

INTRODUCTION

The concept of a type-2 fuzzy set, which is an extension of the concept of an ordinary fuzzy set, was introduced by Zadeh (1975). A type-2 fuzzy set is characterized by a membership function, i.e., the membership value for each element of this set is a fuzzy set in (0, 1), unlike an ordinary fuzzy set where the membership value is a crisp number in (0, 1). Hisdal (1981) discussed the IF THEN ELSE statement and interval-valued fuzzy sets of higher type. Jhon (1998) studied an appraisal of theory and applications on type-2 fuzzy sets. Stephen Dinagar and Anbalagan (2011) presented new ranking function and arithmetic operations on generalized type-2 trapezoidal fuzzy numbers. The fuzzy matrices introduced first time by Thomason (1977), and discussed about the convergence of powers of fuzzy matrix. Kim (1988) presented some important results on determinant of square fuzzy matrices. Ragab and Emam (1995) presented some properties of the min-max composition of fuzzy matrices. Shyamal and Pal (2007) first time introduced triangular fuzzy matrices. Recently Stephen Dinagar and Latha (2012) introduced type-2 triangular fuzzy matrices. In Stephen Dinagar and Latha (2013) presented some types and properties of type-2 triangular fuzzy matrices. The paper is organized as follows. Firstly in section-2 of this paper, we recall the definition of type-2 triangular fuzzy number and some operations on type-2 triangular fuzzy numbers. In section-3, we review the definition of type-2 triangular fuzzy matrices (T2TFM) and some operations on T2TFMs. In section-4, we define adjoint of T2TFMs. In section-5, we derive some more special properties of adjoint of T2TFMs. In section – 6, relevant numerical examples are presented. Finally in section - 7, conclusion is also included.

TYPE-2 TRIANGULAR FUZZY NUMBERS

Definition: Fuzzy set

A fuzzy set is characterized by a membership function mapping the elements of a domain, space or universe of discourse X to the unit interval (0,1).

A fuzzy set A in a universe of discourse X is defined as the following set of pairs:

$$A = \{(x, \mu_A(x)); x \in X\}.$$

Here $\mu_A: X \to (0,1)$ is a mapping called the degree of membership function of the fuzzy set A and $\mu_A(x)$ is called the membership value of $x \in X$ in the fuzzy set A. These membership grades are often represented by real numbers ranging from (0,1).

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Definition: (Zadeh) Type-2 fuzzy set

A type-2 fuzzy set is a fuzzy set whose membership values are fuzzy sets on (0,1).

Definition

The type-2 fuzzy sets are defined by functions of the formμ_A: x → χ ((0,1)) where χ ((0,1)) denotes the set of all ordinary fuzzy sets that can be defined within the universal set (0,1). An example (4) of a membership function of this type is given in Fig.1.

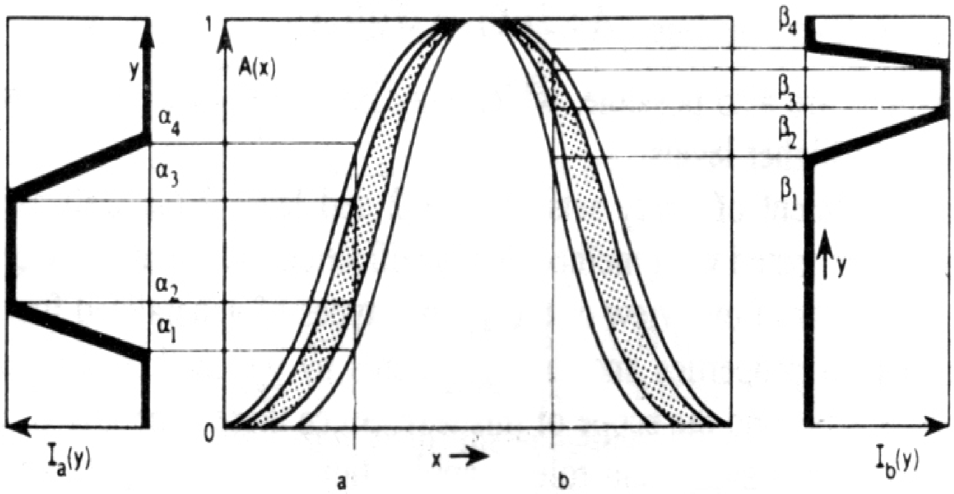

Fig.1. Illustration of the concept of a fuzzy set of type-2

Definition: Type-2 fuzzy number (7)

Let $\tilde{\tilde{A}}$ be a type-2 fuzzy set defined in the universe of discourse R. If the following conditions are satisfied:

(i) $\tilde{\tilde{A}}$ is normal,

(ii) $\tilde{\tilde{A}}$ is a convex set,

(iii) The support of $\tilde{\tilde{A}}$ is closed and bounded, then $\tilde{\tilde{A}}$ is called a type-2 fuzzy number.

Definition: Type-2 triangular fuzzy number

A type-2 triangular fuzzy number $\tilde{\tilde{A}}$ on R is given by $\tilde{\tilde{A}}$ = {(x,($\mu_A{}^1$(x),$\mu_A{}^2$(x),$\mu_A{}^3$(x)); xϵR} and $\mu_A{}^1(x) \leq \mu_A{}^2(x) \leq \mu_A{}^3(x)$, for all xϵ R. Denote $\tilde{\tilde{A}} = (\tilde{A}_1,\tilde{A}_2,\tilde{A}_3)$, where $\tilde{A}_1 = (A_1{}^L,A_1{}^N,A_1{}^U)$, $\tilde{A}_2 = (A_2{}^L,A_2{}^N,A_2{}^U)$ and$\tilde{A}_3 = (A_3{}^L,A_3{}^N,A_3{}^U)$ are same type of fuzzy numbers.

Arithmetic operations on type-2 triangular fuzzy numbers (8)

Let $\tilde{\tilde{a}} = (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3) = ((a_1{}^L,a_1{}^N,a_1{}^U),(a_2{}^L,a_2{}^N,a_2{}^U),(a_3{}^L,a_3{}^N,a_3{}^U))$ and $\tilde{\tilde{b}} = (\tilde{b}_1, \tilde{b}_2, \tilde{b}_3) = ((b_1{}^L,b_1{}^N,b_1{}^U),(b_2{}^L,b_2{}^N,b_2{}^U),(b_3{}^L,b_3{}^N,b_3{}^U))$ be two type-2 triangular fuzzy numbers. Then we define,

(i) Addition:

$\tilde{\tilde{a}} + \tilde{\tilde{b}}$= $((a_1{}^L+b_1{}^L,a_1{}^N+b_1{}^N,a_1{}^U+b_1{}^U),(a_2{}^L+b_2{}^L,a_2{}^N+b_2{}^N,a_2{}^U+b_2{}^U),(a_3{}^L+b_3{}^L,a_3{}^N+b_3{}^N, a_3{}^U+b_3{}^U))$

(ii) Subtraction:

$\tilde{\tilde{a}} - \tilde{\tilde{b}}$= $((a_1{}^L-b_3{}^U,a_1{}^N-b_3{}^N,a_1{}^U-b_3{}^L),(a_2{}^L-b_2{}^U,a_2{}^N-b_2{}^N,a_2{}^U-b_2{}^L),(a_3{}^L-b_1{}^U,a_3{}^N-b_1{}^N, a_3{}^U-b_1{}^L))$

(iii) Scalar multiplication:

If k ≥ 0 and k ϵ R then k$\tilde{\tilde{a}}$ = $((ka_1{}^L,ka_1{}^N,ka_1{}^U),(ka_2{}^L,ka_2{}^N,ka_2{}^U),(ka_3{}^L,ka_3{}^N,ka_3{}^U))$ and
if k < 0 and k ϵ R then k$\tilde{\tilde{a}}$ = $((ka_3{}^U,ka_3{}^N,ka_3{}^L),(ka_2{}^U,ka_2{}^N,ka_2{}^L),(ka_1{}^U,ka_1{}^N,ka_1{}^L))$.

(iv) Multiplication:

Define $\sigma b = b_1^L+b_1^N+b_1^U+b_2^L+b_2^N+b_2^U+b_3^L+b_3^N+b_3^U$.If $\sigma b \geq 0$, then
$\tilde{\tilde{a}} \times \tilde{\tilde{b}} = \left((\frac{a_1{}^L\sigma b}{9}, \frac{a_1{}^N\sigma b}{9}, \frac{a_1{}^U\sigma b}{9}), (\frac{a_2{}^L\sigma b}{9}, \frac{a_2{}^N\sigma b}{9}, \frac{a_2{}^U\sigma b}{9}), (\frac{a_3{}^L\sigma b}{9}, \frac{a_3{}^N\sigma b}{9}, \frac{a_3{}^U\sigma b}{9}) \right)$.
If $\sigma b<0$, then
$\tilde{\tilde{a}} \times \tilde{\tilde{b}} = \left((\frac{a_3{}^U\sigma b}{9}, \frac{a_3{}^N\sigma b}{9}, \frac{a_3{}^L\sigma b}{9}), (\frac{a_2{}^U\sigma b}{9}, \frac{a_2{}^N\sigma b}{9}, \frac{a_2{}^L\sigma b}{9}), (\frac{a_1{}^U\sigma b}{9}, \frac{a_1{}^N\sigma b}{9}, \frac{a_1{}^L\sigma b}{9}) \right)$.

(v) Division:

Whenever$\sigma b \neq 0$ we define division as follows:If $\sigma b>0$, then
$\frac{\tilde{\tilde{a}}}{\tilde{\tilde{b}}} = \left((\frac{9a_1{}^L}{\sigma b}, \frac{9a_1{}^N}{\sigma b}, \frac{9a_1{}^U}{\sigma b}), (\frac{9a_2{}^L}{\sigma b}, \frac{9a_2{}^N}{\sigma b}, \frac{9a_2{}^U}{\sigma b}), (\frac{9a_3{}^L}{\sigma b}, \frac{9a_3{}^N}{\sigma b}, \frac{9a_3{}^U}{\sigma b}) \right)$.
If $\sigma b<0$, then
$\frac{\tilde{\tilde{a}}}{\tilde{\tilde{b}}} = \left((\frac{9a_3{}^U}{\sigma b}, \frac{9a_3{}^N}{\sigma b}, \frac{9a_3{}^L}{\sigma b}), (\frac{9a_2{}^U}{\sigma b}, \frac{9a_2{}^N}{\sigma b}, \frac{9a_2{}^L}{\sigma b}), (\frac{9a_1{}^U}{\sigma b}, \frac{9a_1{}^N}{\sigma b}, \frac{9a_1{}^L}{\sigma b}) \right)$.

The proposed ranking function (8)

Let F(R) be the set of all type-2 normal triangular fuzzy numbers. One convenient approach for solving numerical valued problem is based on the concept of comparison of fuzzy numbers by use of ranking function. An effective approach for ordering the elements of F(R) is to define a linear ranking function Ř:F(R) → Rwhich maps each fuzzy number into R.

Suppose if $\tilde{\tilde{A}} = (\tilde{A}_1, \tilde{A}_2, \tilde{A}_3)$, $= ((A_1^L,A_1^N,A_1^U),(A_2^L,A_2^N,A_2^U),(A_3^L,A_3^N,A_3^U))$ then we define $\check{R}(\tilde{\tilde{A}}) = (A_1^L+A_1^N+A_1^U+A_2^L+A_2^N+A_2^U+A_3^L+A_3^N+A_3^U) / 9$.

Also we define orders on F(R) by
$\check{R}(\tilde{\tilde{A}}) \geq \check{R}(\tilde{\tilde{B}})$ if and only if $\tilde{\tilde{A}} \gtrsim_{\check{R}} \tilde{\tilde{B}}$,
$\check{R}(\tilde{\tilde{A}}) \leq \check{R}(\tilde{\tilde{B}})$ if and only if $\tilde{\tilde{A}} \lesssim_{\check{R}} \tilde{\tilde{B}}$
and $\check{R}(\tilde{\tilde{A}}) = \check{R}(\tilde{\tilde{B}})$ if and only if $\tilde{\tilde{A}} \approx_{\check{R}} \tilde{\tilde{B}}$.

TYPE-2 TRIANGULAR FUZZY MATRICES (T2TFMS) (8)

Definition: Type-2 triangular fuzzy matrix (T2TFM)

A type-2 triangular fuzzy matrix (T2TFM) of order m×n is defined as A = $(\tilde{\tilde{a}}_{ij})_{mxn}$ where the ijth element $\tilde{\tilde{a}}_{ij}$ of A is the type-2 triangular fuzzy number.

3.2. Operations onT2TFMs

As for classical matrices we define the following operations on T2TFMs. Let A = $(\tilde{\tilde{a}}_{ij})$ and B = $(\tilde{\tilde{b}}_{ij})$ be two T2TFMs of same order. Then we have the following:

(i) A+B = $(\tilde{\tilde{a}}_{ij} + \tilde{\tilde{b}}_{ij})$
(ii) A−B = $(\tilde{\tilde{a}}_{ij} - \tilde{\tilde{b}}_{ij})$
(iii) For A = $(\tilde{\tilde{a}}_{ij})_{mxn}$ and B = $(\tilde{\tilde{b}}_{ij})_{nxk}$ then AB = $(\tilde{\tilde{c}}_{ij})_{mxk}$ where $\tilde{\tilde{c}}_{ij} = \sum_{p=1}^{n} \tilde{\tilde{a}}_{ip} . \tilde{\tilde{b}}_{pj}$, i=1,2,…..,m and j=1,2,…..,k.
(iv) A^Tor A′= $(\tilde{\tilde{a}}_{ji})$
(v) kA = $(k\tilde{\tilde{a}}_{ij})$, where k is a scalar.

3.3. Definition: Equal type-2 triangular fuzzy matrices

Two type-2 triangular fuzzy matrices A = $(\tilde{\tilde{a}}_{ij})$ and B = $(\tilde{\tilde{b}}_{ij})$ of the same order are said to be equal if the rank of their elements in the corresponding positions are equal. Also it is denoted by A = B.

Notation

Let A = $(\tilde{\tilde{a}}_{ij})$ be a type-2 triangular fuzzy matrix. Suppose if we take rank for every $\tilde{\tilde{a}}_{ij}$ in A then A is converted into a classical matrix. It is denoted by A = $(\tilde{\tilde{a}}_{ij})$ **c=** $\left(\check{R}(\tilde{\tilde{a}}_{ij})\right)$

4. ADJOINT OF T2TFM

4.1. Definition: Determinant of T2TFM

The determinant of a nxn T2TFM A = $(\tilde{\tilde{a}}_{ij})$ is denoted by | A | or det(A) and is defined as follows:
| A | = $\sum_{p\in S_n} sign\ p\ \prod_{i=1}^{n} \tilde{\tilde{a}}_{ip(i)}$

$$=\sum_{p\in S_n} sign\ p\ \tilde{\tilde{a}}_{1p(1)}\tilde{\tilde{a}}_{2p(2)} \dots \tilde{\tilde{a}}_{np(n)}$$

Where $\tilde{\tilde{a}}_{ip(i)}$ are type-2 triangular fuzzy numbers and S_n denotes the symmetric group of all permutations of the indices {1, 2, 3, . . . , n} and sign p = 1 or -1 according as the permutation p = $\begin{pmatrix} 1 & 2 & \dots\dots & n \\ p(1) & p(2) & \dots\dots & p(n) \end{pmatrix}$ is even or odd respectively.

4.2. Definition: Minor

Let A = $(\tilde{\tilde{a}}_{ij})$ be a square T2TFM of order n. The minor of an element $\tilde{\tilde{a}}_{ij}$ in A is a determinant of order (n – 1) x (n – 1) which is obtained by deleting the i^th row and the j^th column from A and is denoted by $\tilde{\tilde{M}}_{ij}$.

4.3. Definition: Cofactor

Let A = $(\tilde{\tilde{a}}_{ij})$ be a square T2TFM of order n. The cofactor of an element $\tilde{\tilde{a}}_{ij}$ in A is denoted by $\tilde{\tilde{A}}_{ij}$ and is defined as $\tilde{\tilde{A}}_{ij}$ = $(-1)^{i+j}\tilde{\tilde{M}}_{ij}$.

4.4. Definition: Aliter definition for determinant

Alternatively, the determinant of a square T2TFM A = $(\tilde{\tilde{a}}_{ij})$ of order n may be expanded in the form

$$| A | = \sum_{j=1}^{n} \tilde{\tilde{a}}_{ij}\, \tilde{\tilde{A}}_{ij}\ ,\ i \in \{1, 2, \dots , n\}$$

where $\tilde{\tilde{A}}_{ij}$ is the cofactor of $\tilde{\tilde{a}}_{ij}$.

Thus the determinant is the sum of the products of the elements of any row (or column) and the cofactors of the corresponding elements of the same row (or column).

4.5. Definition: Adjoint

Let A = $(\tilde{\tilde{a}}_{ij})$ be a square T2TFM of order n. Find the cofactor $\tilde{\tilde{A}}_{ij}$ of every element $\tilde{\tilde{a}}_{ij}$ in A and replace every $\tilde{\tilde{a}}_{ij}$ by its cofactor $\tilde{\tilde{A}}_{ij}$ in A and let it be B. ie, B = $(\tilde{\tilde{A}}_{ij})$. Then the transpose of B is called the adjoint or adjugate of A and is denoted by adjA. ie, B′ = $(\tilde{\tilde{A}}_{ji})$ = adjA.

5. PROPERTIES OF ADJOINT OF T2TFMs

Property: 1

If A = $(\tilde{\tilde{a}}_{ij})$ is a square T2TFM then A(adjA) is a diagonal-equivalent T2TFM.

Proof:

Let A = $\begin{pmatrix} \tilde{\tilde{a}}_{11} & \tilde{\tilde{a}}_{12} \\ \tilde{\tilde{a}}_{21} & \tilde{\tilde{a}}_{22} \end{pmatrix}$

Then adjA = $\begin{pmatrix} \tilde{\tilde{A}}_{11} & \tilde{\tilde{A}}_{21} \\ \tilde{\tilde{A}}_{12} & \tilde{\tilde{A}}_{22} \end{pmatrix}$

Now A(adjA) = $\begin{pmatrix} \tilde{\tilde{a}}_{11} & \tilde{\tilde{a}}_{12} \\ \tilde{\tilde{a}}_{21} & \tilde{\tilde{a}}_{22} \end{pmatrix}\begin{pmatrix} \tilde{\tilde{A}}_{11} & \tilde{\tilde{A}}_{21} \\ \tilde{\tilde{A}}_{12} & \tilde{\tilde{A}}_{22} \end{pmatrix}$

$$= \begin{pmatrix} \tilde{\tilde{a}}_{11}\tilde{\tilde{A}}_{11} + \tilde{\tilde{a}}_{12}\tilde{\tilde{A}}_{12} & \tilde{\tilde{a}}_{11}\tilde{\tilde{A}}_{21} + \tilde{\tilde{a}}_{12}\tilde{\tilde{A}}_{22} \\ \tilde{\tilde{a}}_{21}\tilde{\tilde{A}}_{11} + \tilde{\tilde{a}}_{22}\tilde{\tilde{A}}_{12} & \tilde{\tilde{a}}_{21}\tilde{\tilde{A}}_{21} + \tilde{\tilde{a}}_{22}\tilde{\tilde{A}}_{22} \end{pmatrix}$$

$$= \begin{pmatrix} \tilde{\tilde{a}}_{11}\tilde{\tilde{A}}_{11} + \tilde{\tilde{a}}_{12}\tilde{\tilde{A}}_{12} & \tilde{\tilde{a}}_{11}(-\tilde{\tilde{a}}_{12}) + \tilde{\tilde{a}}_{12}\tilde{\tilde{a}}_{11} \\ \tilde{\tilde{a}}_{21}\tilde{\tilde{a}}_{22} + \tilde{\tilde{a}}_{22}(-\tilde{\tilde{a}}_{21}) & \tilde{\tilde{a}}_{21}\tilde{\tilde{A}}_{21} + \tilde{\tilde{a}}_{22}\tilde{\tilde{A}}_{22} \end{pmatrix}$$

$$= \begin{pmatrix} |A| & \tilde{\tilde{0}} \\ \tilde{\tilde{0}} & |A| \end{pmatrix}$$

Which is a diagonal-equivalent T2TFM.

Property: 2

If A = $(\tilde{\tilde{a}}_{ij})$ is a square T2TFM then (adjA)A is a diagonal-equivalent T2TFM.

Proof:

Let A = $\begin{pmatrix} \tilde{\tilde{a}}_{11} & \tilde{\tilde{a}}_{12} \\ \tilde{\tilde{a}}_{21} & \tilde{\tilde{a}}_{22} \end{pmatrix}$

Then adjA = $\begin{pmatrix} \tilde{\tilde{A}}_{11} & \tilde{\tilde{A}}_{21} \\ \tilde{\tilde{A}}_{12} & \tilde{\tilde{A}}_{22} \end{pmatrix}$

Now (adjA)A = $\begin{pmatrix} \tilde{\tilde{A}}_{11} & \tilde{\tilde{A}}_{21} \\ \tilde{\tilde{A}}_{12} & \tilde{\tilde{A}}_{22} \end{pmatrix}\begin{pmatrix} \tilde{\tilde{a}}_{11} & \tilde{\tilde{a}}_{12} \\ \tilde{\tilde{a}}_{21} & \tilde{\tilde{a}}_{22} \end{pmatrix}$

$$= \begin{pmatrix} \tilde{\tilde{A}}_{11}\tilde{\tilde{a}}_{11} + \tilde{\tilde{A}}_{21}\tilde{\tilde{a}}_{21} & \tilde{\tilde{A}}_{11}\tilde{\tilde{a}}_{12} + \tilde{\tilde{A}}_{21}\tilde{\tilde{a}}_{22} \\ \tilde{\tilde{A}}_{12}\tilde{\tilde{a}}_{11} + \tilde{\tilde{A}}_{22}\tilde{\tilde{a}}_{21} & \tilde{\tilde{A}}_{12}\tilde{\tilde{a}}_{12} + \tilde{\tilde{A}}_{22}\tilde{\tilde{a}}_{22} \end{pmatrix}$$

$$= \begin{pmatrix} \tilde{\tilde{A}}_{11}\tilde{\tilde{a}}_{11} + \tilde{\tilde{A}}_{21}\tilde{\tilde{a}}_{21} & \tilde{\tilde{a}}_{22}\tilde{\tilde{a}}_{12} + (-\tilde{\tilde{a}}_{12})\tilde{\tilde{a}}_{22} \\ (-\tilde{\tilde{a}}_{21})\tilde{\tilde{a}}_{11} + \tilde{\tilde{a}}_{11}\tilde{\tilde{a}}_{21} & \tilde{\tilde{A}}_{12}\tilde{\tilde{a}}_{12} + \tilde{\tilde{A}}_{22}\tilde{\tilde{a}}_{22} \end{pmatrix}$$

$$= \begin{pmatrix} |A| & \tilde{\tilde{0}} \\ \tilde{\tilde{0}} & |A| \end{pmatrix}$$

Which is a diagonal-equivalent T2TFM.

Property: 3

If A = $(\tilde{\tilde{a}}_{ij})$ is a square T2TFM of order 2, then A(adjA) = (adjA)A c= Ř(| A |) I_2

Proof:

By property:1, we have

A(adjA) = $\begin{pmatrix} |A| & \tilde{\tilde{0}} \\ \tilde{\tilde{0}} & |A| \end{pmatrix}$

c= $\begin{pmatrix} Ř(|A|) & 0 \\ 0 & Ř(|A|) \end{pmatrix}$

= Ř(| A |) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

= Ř(| A |) I_2

i.e, A(adjA) c= Ř(| A |) I_2 $\quad ---\rightarrow$ (5.1)

By property:2, we have

(adjA)A = $\begin{pmatrix} |A| & \tilde{\tilde{0}} \\ \tilde{\tilde{0}} & |A| \end{pmatrix}$

c= $\begin{pmatrix} Ř(|A|) & 0 \\ 0 & Ř(|A|) \end{pmatrix}$

= Ř(| A |) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

= Ř(| A |) I_2

i.e, (adjA)A c= Ř(| A |) I_2 $\quad ---\rightarrow$ (5.2)

From (5.1) and (5.2), we have A(adjA) = (adjA)A c= Ř(| A |) I_2

Property: 4

Let A = $(\tilde{\tilde{a}}_{ij})$ be a square T2TFM of order n. If A contains a row with type-2 zero-equivalent triangular fuzzy numbers then (adjA)A is a null-equivalent T2TFM.

Proof:

Let A = $(\tilde{\tilde{a}}_{ij})$ be a square T2TFM of order n and let B = $(\tilde{\tilde{b}}_{ij})$ = adjA. Then by the definition of adjoint matrix, the ijth element $\tilde{\tilde{b}}_{ij}$ of B is $\tilde{\tilde{A}}_{ji}$, where $\tilde{\tilde{A}}_{ji}$ is the cofactor of $\tilde{\tilde{a}}_{ji}$ in A which is obtained by deleting the jth row and the ith column from A.
Without loss of generality we assume that the rth row of A be $\tilde{\tilde{0}}$. Therefore all the elements $\tilde{\tilde{a}}_{rj} = \tilde{\tilde{0}}$ for all j. We know that if all the elements of a row (or column) of A are $\tilde{\tilde{0}}$ then | A | is also $\tilde{\tilde{0}}$. Hence all the elements of adjA are $\tilde{\tilde{A}}_{ij} = \tilde{\tilde{0}}$ except j≠ r.
Let C = (adjA)A. Then the ijth element $\tilde{\tilde{c}}_{ij}$ of C is

$$\tilde{\tilde{c}}_{ij} = \sum_{k=1}^{n} \tilde{\tilde{A}}_{ik}\, \tilde{\tilde{a}}_{kj}$$
$$= \sum_{k\neq r} \tilde{\tilde{A}}_{ik}\, \tilde{\tilde{a}}_{kj} + \tilde{\tilde{A}}_{ir}\tilde{\tilde{a}}_{rj}$$

Now all $\tilde{\tilde{A}}_{ik} = \tilde{\tilde{0}}$, k≠ r and $\tilde{\tilde{a}}_{rj} = \tilde{\tilde{0}}$. Hence $\tilde{\tilde{c}}_{ij} = \tilde{\tilde{0}}$ for all i, j = 1, 2, ... , n.
Thus C = (adjA)A is a null-equivalent T2TFM.

Property: 5

Let A = $(\tilde{\tilde{a}}_{ij})$ be a square T2TFM of order n. If A contains a column with type-2 zero-equivalent triangular fuzzy numbers then A(adjA) is a null-equivalent T2TFM.

Proof:

Let A = $(\tilde{\tilde{a}}_{ij})$ be a square T2TFM of order n and let B = $(\tilde{\tilde{b}}_{ij})$ = adjA. Then by the definition of adjoint matrix, the ijth element $\tilde{\tilde{b}}_{ij}$ of B is $\tilde{\tilde{A}}_{ji}$, where $\tilde{\tilde{A}}_{ji}$ is the cofactor of $\tilde{\tilde{a}}_{ji}$ in A which is obtained by deleting the jth row and the ith column from A.
Without loss of generality we assume that the rth column of A be $\tilde{\tilde{0}}$. Therefore all the elements $\tilde{\tilde{a}}_{ir} = \tilde{\tilde{0}}$ for all i. We know that if all the elements of a row (or column) of A are $\tilde{\tilde{0}}$ then | A | is also $\tilde{\tilde{0}}$. Hence all the elements of adjA are $\tilde{\tilde{A}}_{ij} = \tilde{\tilde{0}}$ except i≠ r.
Let C = A(adjA). Then the ijth element $\tilde{\tilde{c}}_{ij}$ of C is

$$\tilde{\tilde{c}}_{ij} = \sum_{k=1}^{n} \tilde{\tilde{a}}_{ik}\, \tilde{\tilde{A}}_{kj}$$
$$= \sum_{k\neq r} \tilde{\tilde{a}}_{ik}\, \tilde{\tilde{A}}_{kj} + \tilde{\tilde{a}}_{ir}\tilde{\tilde{A}}_{rj}$$

Now all $\tilde{\tilde{A}}_{kj} = \tilde{\tilde{0}}$, k≠ r and $\tilde{\tilde{a}}_{ir} = \tilde{\tilde{0}}$. Hence $\tilde{\tilde{c}}_{ij} = \tilde{\tilde{0}}$ for all i, j = 1, 2, ... , n.
Thus C = A(adjA) is a null-equivalent T2TFM.

Property: 6

Let A = $(\tilde{\tilde{a}}_{ij})$ be a square T2TFM of order n. If A is symmetric T2TFM then adjA is also symmetric T2TFM.

Proof:

Let A = $(\tilde{\tilde{a}}_{ij})$ be a symmetric T2TFM and let B = $(\tilde{\tilde{b}}_{ij})$ = adjA. Then by the definition of adjoint matrix the ijth element $\tilde{\tilde{b}}_{ij}$ of B is $\tilde{\tilde{A}}_{ji}$, where $\tilde{\tilde{A}}_{ji}$ is the cofactor of $\tilde{\tilde{a}}_{ji}$ in A.
Since A is symmetric T2TFM,

$$\tilde{\tilde{a}}_{ij} = \tilde{\tilde{a}}_{ji} \text{ for all i, j = 1, 2, ... , n.}$$

Hence $\tilde{\tilde{A}}_{ij} = \tilde{\tilde{A}}_{ji}$ for all i, j = 1, 2, ... , n.

ie, $\tilde{\tilde{b}}_{ji} = \tilde{\tilde{b}}_{ij}$ for all i, j = 1, 2, ... , n.

ie, B = $(\tilde{\tilde{b}}_{ij})$ is a symmetric T2TFM.

Property: 7

Let A = $(\tilde{\tilde{a}}_{ij})$ be a square T2TFM of order n. If A is null-equivalent T2TFM then adjA is also null-equivalent T2TFM.

Proof:

Let A = $(\tilde{\tilde{a}}_{ij})$ be a null-equivalent T2TFM and let B = $(\tilde{\tilde{b}}_{ij})$ = adjA. Then $\tilde{\tilde{b}}_{ij} = \tilde{\tilde{A}}_{ji}$, where $\tilde{\tilde{A}}_{ji}$ is the cofactor of $\tilde{\tilde{a}}_{ji}$ in A.
Since A is a null-equivalent T2TFM, all
$\tilde{\tilde{a}}_{ij} = \tilde{\tilde{0}}$ for all i, j = 1, 2, ... , n.
Hence $\tilde{\tilde{A}}_{ij} = \tilde{\tilde{0}}$ for all i, j = 1, 2, ... , n.
ie, $\tilde{\tilde{b}}_{ji} = \tilde{\tilde{0}}$ for all i, j = 1, 2, ... , n.
Therefore B = $(\tilde{\tilde{b}}_{ij})$ = adjA is also null-equivalent T2TFM.

Property: 8

For a square T2TFM of order n, adj(A′) = (adjA)′

Proof:

Let A = $(\tilde{\tilde{a}}_{ij})$ be a square T2TFM of order n. Then by definition adjA = $(\tilde{\tilde{A}}_{ji})$.
Hence (adjA)′ = $(\tilde{\tilde{A}}_{ij})$ $-\,-\,-\;-\,-\,-\,-\,-\,-\,\rightarrow$ (5.3)

Also, A′ = $(\tilde{\tilde{a}}_{ji})$. Now, adj(A′) =$(\tilde{\tilde{A}}_{ij})$ $-\,-\,-\,\rightarrow$ (5.4)

From (5.3) and (5.4), we have adj(A′) = (adjA)′ .

Property: 9

If A is an unit-equivalent T2TFM of order n, then adjA is also an unit-equivalent T2TFM of order n.

Proof:

Let A = $(\tilde{\tilde{a}}_{ij})$ be an unit-equivalent T2TFM of order n. Then all the entries in the principal diagonal $\tilde{\tilde{a}}_{ii}$= $\tilde{\tilde{1}}$ and the remaining entries outside the principal diagonal are $\tilde{\tilde{0}}$.
Hence the cofactor $\tilde{\tilde{A}}_{ij}$ of every entry of A is as in the following manner:
$\tilde{\tilde{A}}_{ij}$= $\tilde{\tilde{0}}$ for all i≠j

and$\tilde{\tilde{A}}_{ii}$= | Î | of order n−1.

We know that | Î | = $\tilde{\tilde{1}}$. Therefore $\tilde{\tilde{A}}_{ii} = \tilde{\tilde{1}}$.
Hence adjA = $(\tilde{\tilde{A}}_{ji})$ is also an unit-equivalent T2TFM of order n.

Property: 10

If A is an unit T2TFM of order n, then adjA is also an unit T2TFM of order n.

Proof:

Let A = $(\tilde{\tilde{a}}_{ij})$ be an unit T2TFM of order n. Then all the entries in the principal diagonal $\tilde{\tilde{a}}_{ii}$= 1 and the remaining entries outside the principal diagonal are 0.
Hence the cofactor $\tilde{\tilde{A}}_{ij}$ of every entry of A is as in the following manner:

$\tilde{\tilde{A}}_{ij}$= 0 for all i≠j
and$\tilde{\tilde{A}}_{ii}$= | I | of order n−1.

We know that | I | = 1. Therefore $\tilde{\tilde{A}}_{ii}$ = 1.
Hence adjA = $(\tilde{\tilde{A}}_{ji})$ is also an unit T2TFM of order n.

6. Numerical example

If A = $\begin{pmatrix} [(-2,-1,3),(-1,0,4),(3,4,8)] & [(-3,-2,2),(-2,-1,3),(2,3,7)] \\ [(0,1,5),(1,2,6),(5,6,10)] & [(1,2,6),(2,3,7),(6,7,11)] \end{pmatrix}$ then

| A | = $[(-10,-5,15),(-5,0,20),(15,20,40)] - [(0,1,5),(1,2,6),(5,6,10)]$
= $[(-20,-11,10),(-11,-2,19),(10,19,40)]$

Now Ř(| A |) = 54 / 9
i.e, Ř(| A |) = 6 .

Also

adjA = $\begin{pmatrix} [(1,2,6),(2,3,7),(6,7,11)] & [(-7,-3,-2),(-3,1,2),(-2,2,3)] \\ [(-10,-6,-5),(-6,-2,-1),(-5,-1,0)] & [(-2,-1,3),(-1,0,4),(3,4,8)] \end{pmatrix}$

A(adjA)=

$$\begin{pmatrix} [(-2,-1,3),(-1,0,4),(3,4,8)] & [(-3,-2,2),(-2,-1,3),(2,3,7)] \\ [(0,1,5),(1,2,6),(5,6,10)] & [(1,2,6),(2,3,7),(6,7,11)] \end{pmatrix} \begin{pmatrix} [(1,2,6),(2,3,7),(6,7,11)] & [(-7,-3,-2),(-3,1,2),(-2,2,3)] \\ [(-10,-6,-5),(-6,-2,-1),(-5,-1,0)] & [(-2,-1,3),(-1,0,4),(3,4,8)] \end{pmatrix}$$

$$= \begin{pmatrix} \begin{bmatrix} [(-10,-5,15),(-5,0,20),(15,20,40)] \\ +[(-28,-12,-8),(-12,4,8),(-8,8,12)] \end{bmatrix} & \begin{bmatrix} [(-8,-4,-3),(-4,0,1),(-3,1,2)] \\ +[(-6,-4,4),(-4,-2,6),(4,6,14)] \end{bmatrix} \\ \begin{bmatrix} [(0,5,25),(5,10,30),(25,30,50)] \\ +[(-44,-28,-24),(-28,-12,-8)] \end{bmatrix} & \begin{bmatrix} [(-10,-6,-5),(-6,-2,-1),(-5,-1,0)] \\ +[(2,4,12),(4,6,14),(12,14,22)] \end{bmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} [(-38,-17,7),(-17,4,28),(7,28,52)] & [(-14,-8,1),(-8,-2,7),(1,7,16)] \\ [(-44,-23,1),(-23,-2,22),(1,22,46)] & [(-8,-2,7),(-2,4,13),(7,13,22)] \end{pmatrix}$$

c= $\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$

= 6 $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

i.e, A(adjA) = Ř(| A |) I_2

(adjA)A=

$$\begin{pmatrix} [(1,2,6),(2,3,7),(6,7,11)] & [(-7,-3,-2),(-3,1,2),(-2,2,3)] \\ [(-10,-6,-5),(-6,-2,-1),(-5,-1,0)] & [(-2,-1,3),(-1,0,4),(3,4,8)] \end{pmatrix} \begin{pmatrix} [(-2,-1,3),(-1,0,4),(3,4,8)] & [(-3,-2,2),(-2,-1,3),(2,3,7)] \\ [(0,1,5),(1,2,6),(5,6,10)] & [(1,2,6),(2,3,7),(6,7,11)] \end{pmatrix}$$

$$= \begin{pmatrix} \begin{bmatrix} [(2,4,12),(4,6,14),(12,14,22)] \\ +[(-28,-12,-8),(-12,4,8),(-8,8,12] \end{bmatrix} & \begin{bmatrix} [(1,2,6),(2,3,7),(6,7,11)] \\ +[(-35,-15,-10),(-15,5,10),(-10,10,15)] \end{bmatrix} \\ \begin{bmatrix} [(-20,-12,-10),(-12,-4,-2),(-10,-2,0)] \\ +[(-8,-4,12),(-4,0,16),(12,16,32)] \end{bmatrix} & \begin{bmatrix} [(-10,-6,-5),(-6,-2,-1),(-5,-1,0)] \\ +[(-10,-5,15),(-5,0,20),(15,20,40)] \end{bmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} [(-26,-8,4),(-8,10,22),(4,22,34)] & [(-34,-13,-4),(-13,8,17),(-4,17,26)] \\ [(-28,-16,2),(-16,-4,14),(2,14,32)] & [(-20,-11,10),(-11,-2,19),(10,19,40)] \end{pmatrix}$$

c= $\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$

= 6 $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

i.e, (adjA)A = Ř(| A |) I_2

Conclusion

In this article adjoint of type-2 triangular fuzzy matrices are defined and also some special properties of adjoint of T2TFMs are proved. Using these results of T2TFMs, some important properties of T2TFMs, involving the notion like inverse of matrix can be studied in future. Also the theories of the discussed T2TFMs may be utilized in further works.

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