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RESEARCH ARTICLE

FUZZY RANDOMLY STOPPED SUM IN LAPLACE TRANSFORM ORDERING

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ABSTRACT

Some new order of stopped sum of independent non-negative fuzzy random variables, when the stopping variable is independent of the summands, is investigated. We show that such fuzzy randomly stopped sums preserve the fuzzy stochastic laplace transform order. For the case of laplace transform orders, there is a suitable converse for each of the order presentation results.

INTRODUCTION

Non-negative random variables and their distributions can be ordered by comparing their properties under various operations. Many stochastic orderings emerge from such comparisons and they are employed extensively in the studies of reliability. Among the frequently discussed stochastic comparisons, the wide applied one is the laplace transform ordering. Laplace transform order compares random variables according to both their location and their spread. Kebir[5] and shaked and shanthikumar[1] have characterized likelihood ration ordering, failure rate ordering, reversed hazard rate ordering and stochastic ordering with the help of laplace transform. Using laplace transform Nanda[8] has characterized the likelihood ratio, failure rate, mean residual life as particular cases. Bengt klefsjo[2] has studied ageing property using laplace transform. Moshe shaked *et.al*.,[6] have studied two notions of stochastic comparisons of non-negative random variables via ratios that are determined by their laplace transform.

In real life situations life and residual life of components are only expressed interms of imprescise quantifications. In such circumstances fuzzy random variables are the fitting tools to capture the random phenomena. Along this line Earnest Lazarus Piriyakumar *et.al*[3] have studied stochastic orderings of fuzzy random variables. Earnest Lazarus Piriyakumar *et.al*[4] have also investigated likelihood ratio orderings and hazard rate orderings of fizzy random variables. The chief goal of this paper is to study stochastic comparisons based on laplace transform order. We introduce the notion of laplace transform order, laplace transform of residual lives order for fuzzy random variables. In real life situations life and residual life components are only expressed interms of imprecise quantities. In such cases stochastic comparisons on fuzzy random variables offer new insights in the study of random phenomenon. Applying 'resolution identity' one can construct a closed fuzzy number from a family of closed intervals. Using this techniques H.C.Wu[10] has constructed the (fuzzy) probability density function of fuzzy random variable from the probability density function. Analogously we have defined (fuzzy) probability distribution function of fuzzy random variables. These theoretical developments have facilitated the results of this paper. Throughout this paper we consider only the kwakernaak's[7] random variables. The organization of the paper is as follows. In section2, the concept of kwakernaak's[7] fuzzy random variable is introduced. We also provide the theoretical formulation of (fuzzy) distribution functions. In section3, we present some comparison of fuzzy random sums in fuzzy laplace transform ordering.

Fuzzy Random Variables

Fuzzy random variables and fuzzy random vectors are the generalization of random variables and random sets[11]. Kwakernaak[7] introduced the concept of a fuzzy random variable as a function $F:\Omega\rightarrow F(R)$ where (Ω, A, P) is a probability space and $F(R)$ denotes the space of upper semi continuous fuzzy sets having compact support and non-empty 1-level set. Puri and Ralescu[9] defined the notion of a fuzzy random variable as a function $F:\Omega\rightarrow F(R^n)$ where (Ω, A, P) is a probability space and $F(R^n)$ denotes all functions $u: R^n\rightarrow[0,1]$ such that $\{x\in R^n; u(x)\geq \alpha\}$ is a non empty and compact for each $\alpha \in (0,1]$. We introduce a notion of a fuzzy random variable slightly different than that of the aforementioned formulations. In this paper fuzzy random variable is defined as a measurable fuzzy set valued function X: Ω→Fo(R) where R is the real line. (Ω, A, P) is a probability space, $F_o(R) = \{A:R\rightarrow [0,1]\}$ and $\{x\in R; A(X) \geq \alpha\}$ is a bounded closed interval for each $\alpha \in (0,1]$. Let x be a non-empty set, P(x) denote the set of all subsets of x, and F(x) denote the set of all fuzzy sets in x. For A$\in F(X)$ we define two subsets of U as follows.

$A_\alpha = \{x \in X; A(X) \geq \alpha\}$ for any $\alpha \in [0,1]$
$A_{\hat{\alpha}} = \{x \in X; A(X) > \alpha\}$ for any $\alpha \in [0,1]$

Where A(x) is the membership function of A.
Let $\{B_\alpha; \alpha \in [0,1]\}$ be a class of subsets of x such that $A_{\hat{\alpha}} c B_{\hat{\alpha}} c A_{\hat{\alpha}}$ for any $\alpha \in [0,1]$. Then A=$\bigcup_{\alpha\in[0,1]} \alpha B_\alpha$
Let R be the real line and (R,B) be the Borel measurable space. Let $F_0(R)$ denote the set of fuzzy subsets A:R→[0,1] with then following properties

(1) $\{x \in R; A(x) = 1\} \neq \phi$
(2) $A_\alpha = \{x \in R; A(x) \geq \alpha\}$ is bounded closed interval in R for each $\alpha \in (0,1]$, i.e. $A_\alpha = [A_\alpha^L, A_\alpha^U] where\ A_\alpha^L = \inf A_\alpha,\ A_\alpha^U = \sup A_\alpha,\ A_\alpha^L, A_\alpha^U \in A_\alpha. -\infty < A_\alpha^L\ and\ A_\alpha^U <+\infty$ for each $\alpha \in (0,1]$, A∈ Fo(R) is called a bounded closed fuzzy number.

Definition

Let (Ω, A, P) be a probability space and U a set valued mapping ε: Ω→I(R)={[x,y]; x,y∈R, x$\leq y\}$ defined as w→ $\varepsilon(w) = [\varepsilon^L(\omega), \varepsilon^U(\omega)]$. Then $\varepsilon(\omega) = [\varepsilon^L(\omega), \varepsilon^U(\omega)]$ is called the random interval if $\varepsilon^L(\omega)$ and $\varepsilon^U(\omega)$ are both random variables on (Ω, A, P). Let X be a real number. Then we can induce a fuzzy number $\tilde{X}$ with the membership function $\mu_{\tilde{X}}$(r)<1; r≠x. We call $\tilde{X}$ as a fuzzy real number induced by the real number X. If $\tilde{X}$ is a canonical fuzzy real number then $\tilde{X}_1^L$=$\tilde{X}_1^U$. Let X be a fuzzy random variable. Then $\tilde{X}_\alpha^L$ and $\tilde{X}_\alpha^U$ are random variables for all α and $\tilde{X}_1^L$=$\tilde{X}_1^U$. For each $\alpha \in$(0,1] we obtain independent identically distributed random variables $\tilde{X}_\alpha^L$ and $\tilde{X}_\alpha^U$ with the same distribution function F(X) for all $\alpha \in (0,1]$. For any fuzzy observation $\tilde{X}$ of the fuzzy random variable $\tilde{X}(\tilde{X}(\omega) = \tilde{X})$. The α-level set $\tilde{X}_\alpha\ is\ \tilde{X}_\alpha = [\tilde{X}_\alpha^L, \tilde{X}_\alpha^U]$.

These $\tilde{X}_\alpha^L$ and $\tilde{X}_\alpha^U$ are observations of $\tilde{X}_\alpha^L$ and $\tilde{X}_\alpha^U$ respectively. Since the fuzzy observation $\tilde{X}$ is a canonical fuzzy number $\tilde{X}_\alpha^L(\omega) = \tilde{X}_\alpha^L$ and $\tilde{X}_\alpha^U(\omega) = \tilde{X}_\alpha^U$ are continuous with respect to α for fixed ω. Thus [$\tilde{X}_\alpha^L$,$\tilde{X}_\alpha^U$] is continuously shrinking w.r.t.α. Since [$\tilde{X}_\alpha^L$,$\tilde{X}_\alpha^U$] is the disjoint union of [$\tilde{X}_\alpha^L$,$\tilde{X}_1^L$] and [$\tilde{X}_1^U$,$\tilde{X}_\alpha^U$] for any real number Xϵ[$\tilde{X}_\alpha^L$,$\tilde{X}_1^U$] we have X=$\tilde{X}_\beta^L$ or X=$\tilde{X}_\beta^U$ for some $\beta \geq \alpha$. Ie., X can be viewed as any one of the end points of the β - level set where $\beta \geq \alpha$. Thus for any x∈ [$\tilde{X}_\alpha^L$,$\tilde{X}_\alpha^U$] we can associate and F($\tilde{X}_\beta^L$) or F($\tilde{X}_\beta^U$) with X. If we construct an interval

$$A_\alpha = [\min \{inf_{\alpha\leq\beta\leq1} F(\tilde{X}_\beta^L), inf_{\alpha\leq\beta\leq1} F(\tilde{X}_\beta^U)\}, \max\{sup_{\alpha\leq\beta\leq1} F(\tilde{X}_\beta^L), sup_{\alpha\leq\beta\leq1} F(\tilde{X}_\beta^U)\}]$$

then this interval will contain all of the α-graded distribution associated with each of x∈[$\tilde{X}_\alpha^L$,$\tilde{X}_\alpha^U$]. We denote by $\tilde{F}(X)$ fuzzy distribution function of the fuzzy random variable X. Then we define the membership function of $\tilde{F}(X)$ for any fixed $\tilde{X}$ by $\mu_{\tilde{F}\tilde{X}}$=$sup_{0\leq r\leq 1}\ \alpha\ 1_{A_\alpha}(r)$ We say that the fuzzy distribution function $\tilde{F}(\tilde{X})$ is induced by the distribution function F(X). Since F(X) is continuous we can rewrite $A_\alpha$as

$$A_\alpha = =[\min\{min_{\alpha\leq\beta\leq1} F(\tilde{X}_\beta^L), min_{\alpha\leq\beta\leq1} F(\tilde{X}_\beta^U)\}, \max\{max_{\alpha\leq\beta\leq1} F(\tilde{X}_\beta^L), max_{\alpha\leq\beta\leq1} F(\tilde{X}_\beta^U)\}].$$

Comparison Of Fuzzy Random Sums In Fuzzy Laplace Transform Ordering

Let X be a non-negative fuzzy random variable with fuzzy distribution function F_X and fuzzy survival function $\overline{F}_X$=1-F_X . Theα-level set of the fuzzy random variable is denoted as X_α=[$\tilde{X}_\alpha^L$, $\tilde{X}_\alpha^U$]; $\alpha \in (0,1]$. The α-level set of the fuzzy distribution function is defined as

$$F_{X_\alpha} = [F_{X_\alpha^L}, F_{X_\beta^U}].$$

Suppose X is a non-negative fuzzy random variable having absolutely continuous fuzzy distribution $F_{X_\alpha}(X)$ with density $F_{X_\alpha}(X)$. Then the ordinary Laplace transform of the density function f_x is given by $L_{X_\alpha}(s) = \int_0^\infty e^{-su} F_{X_\alpha}(u)du, S > 0$. The Laplace transform of F_{X_α} is defined by $L^*_{X_\alpha}(s) = \int_0^\infty e^{-su} F_{X_\alpha}(u)du$. Let $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ be two sequences of fuzzy random variables, hence forth simply denoted by $\{\mathcal{F}_{X_n}\}, \{\mathcal{F}_{Y_n}\}$. Let M,N be non-negative integer valued fuzzy random variables independent of the sequences $\{\mathcal{F}_{X_n}\}, \{\mathcal{F}_{Y_n}\}$.

Increasing Fuzzy Convex Order

Let X and Y be two non-negative real valued fuzzy random variables with respective fuzzy distribution functions F_X and G_Y. The fuzzy random variable X is said to be smaller than Y in the increasing fuzzy convex order, denoted by X $<_{\mathcal{F}_{icx}}$ Y

$\int_x^\infty \{1 - F_{X_\alpha}(u)\}du \leq \int_x^\infty \{1 - G_{Y_\alpha}(u)\}du$, all real x. →(3.1)

ie., X$<_{\mathcal{F}_{icx}}$ Y iff E f(x)≤E f(y), all convex ↑f.

Fuzzy Laplace Domination Order

Let X and Y be two non-negative red valued fuzzy random variables with EX<∞. The fuzzy random variable X is said to be smaller than fuzzy random variable Y in the fuzzy Laplace domination order, denoted by

X$<_{\mathcal{F}_{ldo}}$ Y iff $E(e^{-SY}) \leq E(e^{-SX})$, all S>0. →(3.2)

The fuzzy convex order is the fuzzy stochastic order with equal means X$<_{\mathcal{F}_{cx}}$ Y ⇔ X$<_{\mathcal{F}_{icx}}$ Y and ES=ET<∞.

For X$<_{\mathcal{F}_{icx}}$ Y (X $<_{\mathcal{F}_{cx}}$ Y) restricted to non-negative fuzzy random variables, we may take $x \geq 0$ and 'f' to be convex and non-decreasing negative fuzzy random variables, we may take $x \geq 0$ and 'f' to be convex and non decreasing (convex, respectively) on [0,∞) in (3.1). See Muller and Stoyan (2002) for an overview and some ramifications of the above orderings. In particular, note that the HNBUE(harmonic New Better than used in Expectation) and the class-L properties, familiar in reliability theory(Klefsjo, 1982, 1983) are special cases of increasing fuzzy convex order and fuzzy laplace domination orderings.

Consider the fuzzy random sums $\mathcal{F}_U = \sum_{n=1}^{M} X_n$, $\mathcal{F}_V = \sum_{n=1}^{N} Y_n$ →(3.3) With the convention $\mathcal{F}_U$=0 if M=0 and $\mathcal{F}_V$=0 if N=0. If $<_{\mathcal{F}_{ST}}$ is a given fuzzy stochastic order such that N$<_{\mathcal{F}_{ST}}$ M, $Y_n <_{\mathcal{F}_{ST}} X_n$ all n≥1 and possibly some reasonable conditions on the sequences $\{\mathcal{F}_{X_n}\}$, $\{\mathcal{F}_{Y_n}\}$ implies $\mathcal{F}_V <_{\mathcal{F}_{ST}} \mathcal{F}_U$. Here we say that fuzzy randomly stopped sums of the type in(3.3) posses an external monotonicity property under the fuzzy stochastic order $\mathcal{F}_{ST}$.

In the context of many applications where such fuzzy randomly stopped sums appear, its known that $\{\mathcal{F}_{X_n}\}$, $\{\mathcal{F}_{Y_n}\}$ are sequences of non-negative fuzzy random variables, often with additional structure such as independence, or even the i.i.d. property. In particular, compound geometric distributions which correspond to the fuzzy random variable $\mathcal{F}_U$ in (3.3), when X_n are in i.i.d. and M is geometric, occurs among others in the study of many queueing systems, reliability, stress-strength models, and risk and ruin problems. Fuzzy stochastic order properties of the summands X_n which are preserved by the geometric compound $\mathcal{F}_U$ (with geometrically distributed M) have been investigated by several authors(see szekli, 1995 and Bhattachayer *et.al.*, 2003) Our purpose in this article is to demonstrate two new monotonicity properties of such fuzzy random sums. Specifically, its shown that fuzzy randomly stopped sums of independent sequences preserve the fuzzy laplace order property of the summanals when the stopping times are correspondingly ordered, and infact when the summands constitute i.i.d. sequences, suitable converses also hold. A forthcoming paper by kulik and szokli(2004) on dependence orderings for some functional of multivariate point processes, is conceptually related to the context of this paper and may be of interest to readers of the present article.

Theorem

Consider the fuzzy randomly shopped sum $\mathcal{F}_U$, $\mathcal{F}_V$ in (3.3) , with fuzzy non-negative summands sequences $\{\mathcal{F}_{X_n}\}$, $\{\mathcal{F}_{Y_n}\}$ respectively. When the summands are i.i.d. sequences,

(i) If N$<_{\mathcal{F}_{LT}}$ M , then $Y_1 <_{\mathcal{F}_{LT}} X_1$ implies $\mathcal{F}_V <_{\mathcal{F}_{LT}} \mathcal{F}_U$,

(ii) If M$<_{\mathcal{F}_{LT}}$ N , then $\mathcal{F}_V <_{\mathcal{F}_{LT}} \mathcal{F}_U$ implies $Y_1 <_{\mathcal{F}_{LT}} X_1$,

(iii)If N $\overset{d}{\Leftrightarrow}$ M then, $\mathcal{F}_V <_{\mathcal{F}_{LT}} \mathcal{F}_U$ if and only if $Y_1 <_{\mathcal{F}_{LT}} X_1$.

Proof

Let $\phi_{\mathcal{F}_M}(Z)$:= E(Z^M), 0<Z<1 be the fuzzy probability generating function of M, and $L_{X_n}(S)$ = E($e^{-S X_n}$), S ≥ 0, the fuzzy laplace transform of the non-negative i.i.d. fuzzy random variables X_n, for n ≥ 1. Let $\phi_{\mathcal{F}_N}$ (Z) and $L_{Y_n}(S)$ be defined similarly as the fuzzy probability generating function of N and the fuzzy laplace transform of Y_n. Further when the summands are i.i.d., denote $L_{X_n}(S)$, $L_{Y_n}(S)$ by $L_{\mathcal{F}_X}(S), L_{\mathcal{F}_Y}(S)$.

(i) Since, for the two i.i.d. sequences ,

$X_n <_{\mathcal{F}_{LT}} Y_n$ iff $L_{\mathcal{F}_Y}(S) \geq L_{\mathcal{F}_X}(S)$, **S ≥0.** ⟶ (3.4)

And for the fuzzy shopping variables M and N,

N$<_{\mathcal{F}_{LT}}$ M iff $\phi_{\mathcal{F}_N}$ (Z) = E(Z^N) , 0<Z<1.
= E($e^{(-lns)N}$), S ≡ Z^{-1} > 1 ⟶ (3.5)
≥ E($e^{(-lns)M}$)
≥E(Z^M)
≥$\phi_{\mathcal{F}_M}$ (Z)

We have , for all S ≥0 ,
E($e^{-S\mathcal{F}_V}$) = $\sum_{n=0}^{\infty} P(N = n)\{L_{\mathcal{F}_Y}(S)\}^n$
= $\phi_{\mathcal{F}_N}(L_{\mathcal{F}_Y}(S))$
≥ $\phi_{\mathcal{F}_M}\left(L_{\mathcal{F}_Y}(S)\right)$, by (3.5) ⟶ (3.6)
≥ $\phi_{\mathcal{F}_M}\left(L_{\mathcal{F}_X}(S)\right)$, by (3.4)
≥ E($e^{-S\mathcal{F}_U}$) which proves $\mathcal{F}_V <_{\mathcal{F}_{LT}} \mathcal{F}_U$.

(ii) Suppose $\mathcal{F}_V <_{\mathcal{F}_{LT}} \mathcal{F}_U$

ie., E($e^{-S\mathcal{F}_V}$) ≥ E($e^{-S\mathcal{F}_U}$) , S ≥0.\
Since M$<_{\mathcal{F}_{LT}}$ N iff $\phi_{\mathcal{F}_M}$ (Z) ≥ $\phi_{\mathcal{F}_N}$ (Z) ,0<Z<1.
the two hypothesis together imply, $\phi_{\mathcal{F}_M}\left(L_{\mathcal{F}_Y}(S)\right) \geq \phi_{\mathcal{F}_N}(L_{\mathcal{F}_Y}(S))$
≥ E($e^{-S\mathcal{F}_V}$)
≥ E($e^{-S\mathcal{F}_U}$)
≥$\phi_{\mathcal{F}_N}(L_{\mathcal{F}_X}(S))$, S ≥0.

Which in turn implies $L_{\mathcal{F}_Y}(S) \geq L_{\mathcal{F}_X}(S)$ for all S ≥0 by monotonicity of the probability generating function $\phi_{\mathcal{F}_M}$. Thus Y_1 $<_{\mathcal{F}_{LT}}$ X_1 .

(iii) Since N $\overset{d}{\Leftrightarrow}$ M iff $\phi_{\mathcal{F}_N} = \phi_{\mathcal{F}_M}$ pointwise .

Ie., iff N$<_{\mathcal{F}_{LT}}$ M $<_{\mathcal{F}_{LT}}$ N, the claim follows by combining (i) and (ii).

Theorem

Consider the fuzzy randomly stopped sums $\mathcal{F}_U$, $\mathcal{F}_V$ in (3.3) , with fuzzy non-negative summands sequences $\{\mathcal{F}_{X_n}\}$, $\{\mathcal{F}_{Y_n}\}$ respectively. When the both summands sequences are independent

(i) if $Y_n <_{\mathcal{F}_{LT}} Z <_{\mathcal{F}_{LT}} X_n$ for some z, all n ≥ 1, then N $<_{\mathcal{F}_{LT}}$ M implies$\mathcal{F}_V <_{\mathcal{F}_{LT}} \mathcal{F}_U$
(ii)if $X_n <_{\mathcal{F}_{LT}} Z <_{\mathcal{F}_{LT}} Y_n$ for some z, all n ≥1,and $inf_n\, P(X_n = 0) = 0$, then $\mathcal{F}_V <_{\mathcal{F}_{LT}} \mathcal{F}_U$ implies N$<_{\mathcal{F}_{LT}}$ M .
(iii)if both sequences are i.i.d. , $X_1 \overset{d}{\Leftrightarrow} Y_1$ with common distribution function continuous at zero, then $\mathcal{F}_V <_{\mathcal{F}_{LT}} \mathcal{F}_U$ if and only if N$<_{\mathcal{F}_{LT}}$ M

Proof

(i) Under the stated hypothesis, we have for S ≥0,

$$E(e^{-S\mathcal{F}_V}) = P(N=0) + \sum_{n=1}^{\infty} P(N=n) E(e^{-S\sum_{i=1}^{n} Y_i} \mid N=n) \quad \longrightarrow (3.7)$$
$$= P(N=0) + \sum_{n=1}^{\infty} P(N=n) \prod_{i=1}^{n} L_{\mathcal{F}_{Y_i}}(S)$$
$$\geq P(N=0) + \sum_{n=1}^{\infty} P(N=n) L^n{}_{\mathcal{F}_Z}(S)$$
$$\geq \phi_{\mathcal{F}_N}(L_{\mathcal{F}_N}(S)), \quad \longrightarrow (3.8)$$

where $L_{\mathcal{F}_Z}(\,.\,)$ is the fuzzy laplace transform of Z.

We have used the independence of N and $\{\mathcal{F}_{Y_i}\}$ in the second step above, and the next step follows from $Y_i <_{\mathcal{F}_{LT}} Z$ all $i \geq 1$. An analogous computation similarly yields,

$$E(e^{-S\mathcal{F}_U}) = P(M=0) + \sum_{n=1}^{\infty} P(M=n) \prod_{i=1}^{n} L_{\mathcal{F}_{X_i}}(S)$$
$$\leq P(M=0) + \sum_{n=1}^{\infty} P(M=n) L^n{}_{\mathcal{F}_Z}(S), \quad \text{Since } Z <_{\mathcal{F}_{LT}} X_i \text{ all } i \geq 1$$
$$\leq \phi_{\mathcal{F}_M}(L_{\mathcal{F}_Z}(S)),$$
$$\leq \phi_{\mathcal{F}_Z}(L_{\mathcal{F}_Z}(S)),$$
$$\leq E(e^{-S\mathcal{F}_V}), \text{ using (3.8)}$$

Note, the second inequality above follows from $N <_{\mathcal{F}_{LT}} M$ and $L_{\mathcal{F}_Z}(S) \in (0,1]$ for S ≥0 .

(ii) Since $\mathcal{F}_V <_{\mathcal{F}_{LT}} \mathcal{F}_U$, using (3.7) its counterpart for $E(e^{-S\mathcal{F}_U})$ and hypothesis $\mathbf{X}_n <_{\mathcal{F}_{LT}} Z <_{\mathcal{F}_{LT}} Y_n$, $n \geq 1$.

We have ,

$$\phi_{\mathcal{F}_N}(L_{\mathcal{F}_Z}(S)) = P(N=0) + \sum_{n=1}^{\infty} P(N=n) L^n{}_{\mathcal{F}_Z}(S)$$
$$\geq P(N=0) + \sum_{n=1}^{\infty} P(N=n) \prod_{i=1}^{n} L_{\mathcal{F}_{Y_i}}(S)$$
$$= E(e^{-S\mathcal{F}_V})$$
$$\geq E(e^{-S\mathcal{F}_U})$$
$$= P(M=0) + \sum_{n=1}^{\infty} P(M=n) \prod_{i=1}^{n} L_{\mathcal{F}_{X_i}}(S)$$
$$\geq P(M=0) + \sum_{n=1}^{\infty} P(M=n) L^n{}_{\mathcal{F}_Z}(S)$$
$$= \phi_{\mathcal{F}_M}\left(L_{\mathcal{F}_Z}(S)\right), \text{for all } S \geq 0. \quad \longrightarrow (3.9)$$

Since $Z := L_{\mathcal{F}_Z}(S) \downarrow$ from 1 to P(Z=0) as S↑ on [0,∞) ; (3.9) is equivalent to $\phi_{\mathcal{F}_N}(Z) \geq \phi_{\mathcal{F}_M}(Z)$ for $Z \in$ (P(Z=0),1] . This is equivalent to $N <_{\mathcal{F}_{LT}} M$, provided P(Z=0) =0. To check that such is indeed the case, let S→∞ in the inequality $E(e^{-SZ}) \leq E(e^{-SX_n})$,S ≥0 which holds in virtue of $X_n <_{\mathcal{F}_{LT}} Z$,all n ≥ 1 , and hence implies P(Z=0) $\leq inf_{n\geq 1} P(X_n = 0) = 0$. (iii)When both $\{\mathcal{F}_{X_n}\}$, $\{\mathcal{F}_{Y_n}\}$ are i.i.d. with $X_1 \overset{d}{\Leftrightarrow} Y_1$, and their common distribution function has no atom at zero; both the hypothesis in (i) - (ii) are satisfied by any fuzzy random variable Z such that $Z \overset{d}{\Leftrightarrow} Y_1$. Hence combining (i) and (ii) implies the desired conclusion.

Theorem

Consider the fuzzy randomly shopped sum $\mathcal{F}_U$, $\mathcal{F}_V$ in (3.3) , with fuzzy non-negative summands sequences $\{\mathcal{F}_{X_n}\}$, $\{\mathcal{F}_{Y_n}\}$ respectively. For i.i.d. Summands, theorem (3.4) (i) – (ii)
Can be improved as follows ,

(i)) If $Y_1 <_{\mathcal{F}_{LT}} X_1$,then $N <_{\mathcal{F}_{LT}} M$ implies $\mathcal{F}_V <_{\mathcal{F}_{LT}} \mathcal{F}_U$,
(ii) $X_1 <_{\mathcal{F}_{LT}} Y_1$ and $P(Y_1 = 0) = 0$ then $\mathcal{F}_V <_{\mathcal{F}_{LT}} \mathcal{F}_U$ if and only if $N <_{\mathcal{F}_{LT}} M$.

Proof

(i) The claim that $Y_1 <_{\mathcal{F}_{LT}} X_1$,and $N <_{\mathcal{F}_{LT}} M$ together imply $\mathcal{F}_V <_{\mathcal{F}_{LT}} \mathcal{F}_U$, is a restatement of theorem (3.3)(ii).
An alternative arguments follows by choosing $Z = Y_1 = Y_i <_{\mathcal{F}_{LT}} X_i$, all i ≥1 in theorem (3.4)(i).

(ii) Suppose $\mathcal{F}_V <_{\mathcal{F}_{LT}} \mathcal{F}_U$. Since each summand sequence is i.i.d. , we have $\phi_{\mathcal{F}_N}(L_{\mathcal{F}_Y}(S)) = E(e^{-S\mathcal{F}_V})$

$$\geq E(e^{-S\mathcal{F}_U})$$
$$= \phi_{\mathcal{F}_M}\left(L_{\mathcal{F}_X}(S)\right)$$
$$\geq \phi_{\mathcal{F}_M}\left(L_{\mathcal{F}_Y}(S)\right),\ S \geq 0. \qquad \longrightarrow (3.10)$$

And similarly, extending the inequality for the first term in the other direction,

$$\phi_{\mathcal{F}_N}(L_{\mathcal{F}_X}(S)) \geq \phi_{\mathcal{F}_N}\left(L_{\mathcal{F}_Y}(S)\right),$$
$$\geq \phi_{\mathcal{F}_M}\left(L_{\mathcal{F}_X}(S)\right),\ S \geq 0. \qquad \longrightarrow (3.11)$$

As $S \rightarrow 0+$, the inequalities (3.10) – (3.11) together
imply $\phi_{\mathcal{F}_N}(Z) \geq \phi_{\mathcal{F}_M}(Z)$, for $A < Z \leq 1$. $\longrightarrow$ (3.12)
Where $A := \min(P(X = 0), P(Y = 0)) = P(Y = 0)$, since $X <_{\mathcal{F}_{LT}} Y$ guarantees

$$P(X = 0) = \lim_{S\to 0+} E(e^{-SX})$$
$$\geq \lim_{S\to 0+} E(e^{-SY})$$
$$= P(Y = 0).$$

If $P(Y = 0) = 0$, then (3.12) implies $N <_{\mathcal{F}_{LT}} M$.
Note that d.f. of X need not be continuous at zero for the last conclusion, although such continuity would be sufficient to conclude $N <_{\mathcal{F}_{LT}} M$.

Corollary

For the fuzzy random sum of non-negative independent fuzzy random variables in (3.3),
(i)) If $Y_n <_{\mathcal{F}_{icx}} X_n$,$n \geq 1$ then $N <_{\mathcal{F}_{LT}} M$ implies $\mathcal{F}_V <_{\mathcal{F}_{LT}} \mathcal{F}_U$,
(ii) if $X_n <_{\mathcal{F}_{icx}} Y_n$, $n \geq 1$, then $\mathcal{F}_V <_{\mathcal{F}_{LT}} \mathcal{F}_U$ if and only if $N <_{\mathcal{F}_{LT}} M$.
(iii)if $X_n \overset{d}{\Leftrightarrow} Y_n$, $n \geq 1$, $\inf_{n\geq 1} P(X_n = 0) = 0$, then $\mathcal{F}_V <_{\mathcal{F}_{LT}} \mathcal{F}_U$ if and only if $N <_{\mathcal{F}_{LT}} M$.

Proof

(i)-(ii) $Y_n <_{\mathcal{F}_{icx}} X_n$ implies that there exists a fuzzy random variable Z_n
Such that $\mathbf{X}_n <_{\mathcal{F}_{ST}} Z_n <_{\mathcal{F}_{cx}} Y_n$ by a result of Mülller and Stoyan (2002, theorem 1.5.14, P22).
Since $<_{\mathcal{F}_{ST}}$ and $<_{\mathcal{F}_{cx}}$ are both stronger than $<_{\mathcal{F}_{LT}}$.
(Viz., $<_{\mathcal{F}_{ST}} \Rightarrow <_{\mathcal{F}_{icx}} \Rightarrow <_{\mathcal{F}_{LT}}$ and $<_{\mathcal{F}_{cx}} \Rightarrow <_{\mathcal{F}_{icx}}$)
We have , $\mathbf{X}_n <_{\mathcal{F}_{LT}} Z_n <_{\mathcal{F}_{LT}} Y_n$.

Now, apply theorem (3.4)(i). This proves the first claim. An analogous argument applies for claim (ii).

(iii) If $X_n \overset{d}{\Leftrightarrow} Y_n$, then $Y_n <_{\mathcal{F}_{icx}} X_n <_{\mathcal{F}_{icx}} Y_n$.

If additionally, $\inf_n P(X_n = 0) = 0$, then the hypothesis in both the claim (i) and (ii) hold.

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