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RESEARCH ARTICLE

WAVELET APPROXIMATION AND ANALYSIS

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ABSTRACT

Wavelets are new families of basis functions that practical measurements of real phenomena require time and resources, they provide not all values but only a finite sequence of values called a Sample of the function representing the phenomenon under consideration. Therefore, the first in the analysis of a data with wavelets consists in approximating its function by means of the sample alone. One of the simplest methods of approximation uses a horizontal stair step extended through each sample point. The resulting steps form a new function denoted here by $\tilde{f}$ and called a simple function or step function, which approximates the sampled function *f*. Although approximations more accurate than simple step exist, they demand more sophisticated Mathematics, so this paper presents to simple steps. A precise notation will prove useful to indicate the location of such steps. In this paper we have used simple steps, Haar wavelets for approximating the different class of data and analyzed by using wavelets.

INTRODUCTION

This paper explains the nature of the Approximation and Analysis of (sample) functions and an algorithm to compute a fast wavelet transform such wavelets have been called Haars Wavelets (1910) [1]. To analyze and synthesize a data which can be any array of data in terms of sample functions (simple wavelets), this paper employs shifts and dilations of Mathematical functions. The first step in applying wavelets to any data or physical phenomenon consists in representing the data under consideration by a Mathematical function *f*. The usefulness of Mathematical functions lies in their efficiency and versatility in representing various types of data or phenomena. For instance the horizontal axis may corresponds to time, while the vertical axis may correspond to the intensity of a data , suppose a sound, the values measure the sound at each time at a fixed location. Alternatively the horizontal axis may correspond to a spatial dimension and then the values measure the intensity of the sound at each location at a common time, similarly the same function *f* may represent the intensity of light along a cross section of an image. In any event, because the same type of Mathematical function *f* can represent many types of data's or phenomena, the same type of analysis or synthesis of *f*, in terms of wavelets or otherwise, in this we apply to all the datas or phenomena, represented by *f*.

Basic unit step functions

For all numbers *a* and *b*, then notation *[a,b)* and is defined the intervals $[a,b)(k)=\{k: a\leq k<b\}$

The analysis of the approximating function $\tilde{f}$ in terms of wavelets requires a precise labeling of each step, by means of shifts and dilations of the basic unit step function denoted by $\varphi[0,1)$ [2].
The unit step function $\varphi[0,1)$ has the values.

$$\varphi\,[0,1)\,(k)=\begin{cases}1\ if\ 0\leq k<1\\ 0\ , Otherwise\end{cases}$$

This is known as unit step function For a step at the same unit height 1 but with a narrower width *b*, this shows the step function $\varphi[0,b)$ denoted by

$$\varphi\,[0,b)\,(k)=\begin{cases}1\ if\ 0\leq k<b\\ 0\ , Otherwise\end{cases}$$

Similarly, for a step at the same unit height 1, but starting at a different location *a* instead of *0* shows the step function $\varphi[a,b)$, defined by

$$\varphi\,[a,b)(k)=\begin{cases}1\ if\ a\leq k<b\\ 0\ , Otherwise\end{cases}$$

Lastly, we to construct a step function at a different height *C*, starting at the location *a* and ending at *b*, then shows that $C\,\varphi[a,b)$, a scalar multiple by *C* of the function $\varphi[a,b)$, so that

$$C\,\varphi\,[a,b)(k)=\begin{cases}C\ if\ a\leq k<b\\ 0\ , Otherwise\end{cases}$$

Approximation of Simple Wavelets

The Haar wavelet transformations express the approximating function $\tilde{f}$ with wavelets by replacing an adjacent (ordinates) pair of steps by one wider step (scaling function) and one wavelet [3].

Wider Step (Scaling) Function

The sum of two adjacent steps with width ½ produces the basic unit step function $\varphi[0,1)$ as defined by
$\varphi\,[0,1) = \varphi[0,1/2) + \varphi[1/2,1)$

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Basic Wavelet

The difference of two narrower steps (adjacent pair) is known as Basic wavelet, denoted by $\Psi[0,1)$ is defined by

$$\Psi[0,1) = \varphi[0,1/2) - \varphi[1/2,1)$$

Adding and Substracting of these two definitions just we obtained the Inverse relation, which expresses as the narrower steps $\varphi[0, 1/2)$ and $\varphi[1/2, 1)$ in terms of the basic unit step $\varphi[0, 1)$ and wavelet $\Psi[0, 1)$, as given by

$$½(\varphi[0, 1) + \Psi[0, 1)) = \varphi[0, 1/2)$$

$$½(\varphi[0, 1) - \Psi[0, 1)) = \varphi[1/2, 1)$$

Shifts and Dilations of the Basic Haar Wavelet

Suppose we have consider more then 2 sample values. To apply the basic approximation starting at a different location a instead of 0, and over the interval extending to b instead of 1, define the shifted and dilated wavelet $\Psi[a, b)$ by the mid point: $(a+b)/2$

$$\Psi[a, b)(k) = \begin{cases} 1 \; if \; a \le k < a/b \\ -1 \; if \; \frac{a}{b} \le k < b \end{cases}$$

We now the definition of scaling function and wavelet is defined as

$$\varphi[a, b) = \varphi[a, a/b) + \varphi[a/b, b)$$

$$\Psi[a, b) = \varphi[a, a/b) - \varphi[a/b, b)$$

Also, Adding and Subtracting of these two definitions just we obtained the Inverse relation,

$$½(\varphi[a, b) + \Psi[a, b) = \varphi[a, a/b)$$

$$½(\varphi[a, b) - \Psi[a, b)) = \varphi[a/b, b)$$

The Ordered Fast Haar Wavelet Approximation

To analyze a data or function in terms of wavelet approximating, the Fast Haar Wavelet Approximating begins with the initialization of an array with 2^n entries and then proceeds with n iterations of the basic approximating explained in the preceding section. For each index $l \epsilon \{1 \dots\dots n\}$ before iteration number l, the array will consist of $2^{n-(l-1)}$ coefficients of $2^{n-(l-1)}$ step functions $\varphi_r^{n-(l-1)}$ defined below. After iteration number l the array will consist of half as many 2^{n-l}, coefficients of 2^{n-l} step functions $\varphi_r^{(n-l)}$ and 2^{n-l} coefficients of wavelets Ψ_r^{n-l}.

Let positive integer n and index $l \in \{0 \dots\dots n\}$, define the step functions $\varphi_r^{(n-l)}$ and wavelets Ψ_r^{n-l} now

$$\varphi_r^{(n-l)}(k) = \varphi[0,1)(2^{n-l}[k-r2^{l-n}])$$

$$= \begin{cases} 1 \; if \; r2^{n-l} \le k < (r+1)2^{n-l} \\ 0 \quad , \qquad Otherwise \end{cases}$$

$$\Psi_r^{n-l}(k) = \Psi[0, 1)(2^{n-l}[k-r2^{l-n}])$$

$$= \begin{cases} 1 \; if \; r2^{n-l} \le k < \left[r + \left(\frac{1}{2}\right)\right] 2^{n-l} \\ -1 \; if \; \left[r + \left(\frac{1}{2}\right)\right] 2^{n-l} \le k < (r+1)2^{n-l} \\ 0 \; , \qquad Otherwise \end{cases}$$

In the forgoing definition the frequency increases with the index n, as in references (2) and (3) By contrast, in such references as (4) and (5), the frequency decreases as the index increases.

The In-place Fast Haar Wavelet Approximation

For each pair $(a_{2r}^{(n-[l-1])}, a_{2r+1}^{(n-[l-1])})$ instead of placing its results in two additional array, the l^{th} sweep of the in-place approximation merely replaces the pair $(a_{2r}^{(n-[l-1])}, a_{2r+1}^{(n-[l-1])})$ *by the new entries* $(a_r^{(n-l)}, c_r^{(n-l)})$.

The In-place Fast Inverse Haar Wavelet Approximation

The Fast Haar Wavelet Approximation neither alters nor diminishes the information contained in the initial array $\vec{y} = (y_0, \dots\dots, y_{2^n-1})$, because each basic approximation

$$a_r^{(l)} = (½)(a_{2r}^{(l-1)} + a_{2r+1}^{(l-1)})$$

$$c_r^{(l)} = (½)(a_{2r}^{(l-1)} - a_{2r+1}^{(l-1)})$$

admits an inverse approximation

$$a_{2r}^{(l-1)} = a_r^{(l)} + c_r^{(l)},$$

$$a_{2r+1}^{(l-1)} = a_r^{(l)} - c_r^{(l)},$$

Repeated applications of the basic inverse approximation just given, beginning with the wavelet coefficients

$$\vec{y}^{(0)} = (a_0^{(n)}, c_0^{(1)}, \dots, c_{2^{(n-1)}-1}^{(1)})$$

Reconstruct the initial array

$$\vec{y}^{(n)} = \vec{y} = (y_0, \dots\dots, y_{2^n-1}).$$

Methods of Approximation

In general, thus, if a sample points (x_j, y_j) includes a value $y_j = f(x_j)$ at height y_j and at time or location x_j, then that sample point corresponds to the step function [3-6].

$$y_j \, \varphi[x_j, x_{j+1})$$

Which approximates f *at* height y_j on the interval $[x_j, x_{j+1})$.

Adding all the step functions corresponding to all the points in the sample yields a formula approximating the simple step function is given by

$$\tilde{f} = y_0 \, \varphi[x_0, x_1) + y_1 \, \varphi[x_1, x_2) + \dots\dots + y_{n-1} \, \varphi[x_{n-1}, x_n)$$

$$\tilde{f} = \sum_{j=0}^{n-1} y_j \, \varphi[x_j, x_{(j+1)})$$

For two adjacent steps at heights y_0 and y_1, the equations just derived yield the following representation with one wider step & one wavelet.

$$\tilde{f} = y_0 \, \varphi[0, 1/2) + y_1 \, \varphi[1/2, 1)$$

$\tilde{f}$ = y_0 ½(φ[0, 1) + Ψ[0, 1)) + y_1 ½(φ[0, 1) - Ψ[0, 1))

= $(y_0+y_1)/2$ φ[0, 1) + $(y_0-y_1)/2$ Ψ[0, 1)

The shift and dilated basic approximation just described applies to all the consecutive pairs of values, separated here by semicolons for convenience in a sample with *2n* values

$y_0, y_1, y_2, \ldots, y_{2r}, y_{2r+1}, \ldots, y_{2(n-1)}, y_{2n-1}$

For Haars wavelets, the initialization consist only in establishing a one dimensional array $\vec{a}(n)$ also called a vector or a finite sequence of sample values of the form

$$\vec{a}^{(n)} = (a_0^{(n)}, a_1^{(n)}, \ldots, a_j^{(n)}, \ldots, a_{2^n-2}^{(n)}, a_{2^n-1}^{(n)})$$

$$y = (y_0, y_1, y_2, \ldots, y_j, \ldots, y_{2^n-2}, y_{2^n-1})$$

A total number of sample values equal to an integral power of two, 2^n, as indicated by the superscript $^{(n)}$. Though indices ranging from 1 through 2^n would also serve the same purpose, indices ranging from 0 though 2^n-1 will accommodate a binary encoding with only binary digits, and will also offer notational simplification in the sampled step function

$$\tilde{f}^{(n)} = \sum_{j=0}^{2n-1} \boldsymbol{a_j^{(n)}} \boldsymbol{\varphi_j^{(n)}}$$

In general, the l^{th} sweep of the basic approximation begins with an array of $2^{n-(l-1)}$ values

$$\vec{a}^{(n-[l-1])} = (a_0^{(n-[l-1])}, \ldots, a_{2^{n-(l-1)}-1}^{(n-[l-1])})$$

And applies the basic approximation to each pairs $(a_{2r}^{(n-[l-1])}, a_{2r+1}^{(n-[l-1])})$ which gives two new wavelet coefficients

$$a_r^{(n-l)} = (a_{2r}^{(n-[l-1])} + a_{2r+1}^{(n-[l-1])})/2$$

$$c_r^{(n-l)} = (a_{2r}^{(n-[l-1])} - a_{2r+1}^{(n-[l-1])})/2$$

These $2^{(n-l)}$ pairs of new coefficients represent the result of the l^{th} sweep, a result that can also be reassembled in to two arrays.

$$\vec{a}^{(n-l)} = (a_0^{(n-l)}, a_1^{(n-l)}, \ldots, a_r^{(n-l)}, \ldots, a_{2^{n-l}-1}^{(n-l)})$$

$$\vec{c}^{(n-l)} = (c_0^{(n-l)}, c_1^{(n-l)}, \ldots, c_r^{(n-l)}, \ldots, c_{2^{n-l}-1}^{(n-l)})$$

The arrays related to the l^{th} sweep have the following significance $\vec{a}^{[n-(l-1)]}$ the beginning array

The values $a_r^{(n-[l-1])}$ of a simple step function $\tilde{f}^{(n-[l-1])}$ that approximates the initial function *f* with $2^{n-(l-1)}$ steps of narrower width $2^{(l-1)-n}$

$$\tilde{f}^{(n-[l-1])} = \sum_{j=0}^{2^{n-(l-1)}-1} \boldsymbol{a_j^{(n-[l-1])}} \boldsymbol{\varphi_j^{(n-[l-1])}}$$

$\vec{a}^{(n-l)}$: The first array produced by the l^{th} sweep

$$\vec{a}^{(n-l)} = (a_0^{(n-l)}, a_1^{(n-l)}, \ldots, a_{2^{n-l}-1}^{(n-l)})$$

The values $a_r^{(n-l)}$ of a simple step function $\tilde{f}^{(n-l)}$ that approximates the initial function f with 2^{n-l} steps of wider width 2^{l-n}

$$\tilde{f}^{(n-l)} = \sum_{j=0}^{2^{(n-1)}-1} \boldsymbol{a_j^{(n-l)}} \boldsymbol{\varphi_j^{(n-l)}}$$

$\vec{c}^{(n-l)}$ the second array produced by the l^{th} sweep

$$\vec{c}^{(n-l)} = (c_0^{(n-l)}, c_1^{(n-l)}, \ldots, c_{2^{n-l}-1}^{(n-l)})$$

The coefficients $c_r^{(n-l)}$ of simple wavelets $\Psi_j^{(n-l)}$ also of wider width 2^{l-n},

$$\dot{f}^{(n-l)} = \sum_{j=0}^{2^{(n-l)}-1} \boldsymbol{c_j^{(n-l)}} \boldsymbol{\Psi_j^{(n-l)}}$$

Thus, the sum of these two array produced by the initial approximation $\tilde{f}^{(n-[l-1])}$ still equals the sum of the two new approximations, $\tilde{f}^{(n-l)}$ and $\dot{f}^{(n-l)}$

$$\tilde{f}^{(n-[l-1])} = \tilde{f}^{(n-l)} + \dot{f}^{(n-l)}$$

In general we consider the pairs $(a_{2r}^{(n-[l-1])}, a_{2r+1}^{(n-[l-1])})$ perform the basic approximation

$$a_r^{(n-l)} = (a_{2r}^{(n-[l-1])} + a_{2r+1}^{(n-[l-1])})/2$$

$$c_r^{(n-l)} = (a_{2r}^{(n-[l-1])} - a_{2r+1}^{(n-[l-1])})/2$$

Replace the initial pair $(a_{2r}^{(n-[l-1])}, a_{2r+1}^{(n-[l-1])})$ by the approximation pair

$(a_r^{(n-l)}, c_r^{(n-l)})$ This pairs is known as In-place basic sweep. The in-place sweep explained in the preceding subsection extends to a complete algorithm mere record-keeping the first few sweeps proceed as follows

Consider,

$$\vec{y}^{(n)} = \vec{y} = (y_0, y_1, y_2, \ldots y_{2r}, y_{2r+1}, \ldots y_{2^n-2}, y_{2^n-1})$$

First Sweep:

$$\vec{y}^{(n-1)} = [(y_0+y_1)/2, (y_0-y_1)/2, \ldots, (y_{2r}+y_{2r+1})/2, (y_{2r}-y_{2r+1})/2, \ldots (y_{2^n-2}+y_{2^n-1})/2, (y_{2^n-2}-y_{2^n-1})/2]$$

$$= (a_0^{(n-1)}, c_0^{(n-1)}, a_1^{(n-1)}, c_1^{(n-1)}, a_2^{(n-1)}, c_2^{(n-1)}, \ldots, a_r^{(n-1)}, c_r^{(n-1)}, \ldots, a_{2^{n-1}-1}^{(n-1)}, c_{2^{n-1}-1}^{(n-1)})$$

Second Sweep

In the new array $\vec{y}^{(n-1)}$ keep but skip over the wavelet coefficient $c_r^{(n-l)}$, and perform a basic sweep on the array $\vec{a}^{(n-l)}$ at its new location, now occupying every other entry in $\vec{y}^{(n-1)}$

$\vec{y}^{(n-2)}$=[$(a_0^{(n-1)}+a_1^{(n-1)})/2$, $c_0^{(n-l)}$, $(a_0^{(n-1)}-a_1^{(n-1)})/2$, $c_1^{(n-l)}$,…….

$(a_{2r}^{(n-1)}+a_{2r+1}^{(n-1)})/2$, $c_{2r}^{(n-l)}$, $(a_{2r}^{(n-1)}+a_{2r+1}^{(n-1)})/2$, $c_{2r+1}^{(n-l)}$

,……..,$(a_{2^{n-1}-2}^{n-1}+a_{2^{n-1}-1}^{n-1})/2$, $c_{2^{n-1}-2}^{n-1}$,

$(a_{2^{n-1}-2}^{n-1}-a_{2^{n-1}-1}^{n-1})/2$, $c_{2^{n-1}-1}^{n-1}$]

=$(a_0^{(n-2)}, c_0^{(n-l)}, c_0^{(n-2)}, c_1^{(n-l)}, a_1^{(n-2)}, c_2^{(n-l)},$

$c_1^{(n-2)}, c_3^{(n-l)}, a_2^{(n-2)}, c_4^{(n-l)},$

$c_2^{(n-2)}, c_5^{(n-l)}$,…….., $c_{2^{n-2}-1}^{(n-2)}, c_{2^{n-1}-1}^{(n-1)})$

In general, the in-place l^{th} sweep begins with an array

$\vec{y}^{(n-[l-1])}$ = $[a_0^{(n-[l-1)}$, $c_0^{(n-l)}, c_0^{(n-2)}, c_1^{(n-l)}, c_0^{(n-3)}, c_2^{(n-l)}, c_1^{(n-2)}$, $c_3^{(n-l)}$,…….

$c_{2^{n-2}-1}^{(n-2)}, c_{2^{n-1}-1}^{(n-1)}]$

Inverse of this method

The in-place Fast Inverse Haar Wavelet Approximation is given, beginning with the wavelet coefficients

$$\vec{y}^{(0)} = (a_0^{(n)}, c_0^{(1)}, \ldots\ldots, c_{2(n-1)-1}^{(1)})$$

Reconstruct the initial array $\vec{y}^{(n)} = \vec{y}$ = (y_0, ……. , y_{2^n-1}).

Implementation

Let a sample points (x_0 , y_0)= (0 , 9) and (x_1 , y_1)=(1/2 , 1) then we find that sample points corresponds to the step function and j=0,1.

$$\tilde{f} = \sum_{j=0}^{n-1} y_j\, \varphi[x_j, x_{(j+1)})$$

$= 9\,\varphi[0, 1/2) + 1\,\varphi[1/2, 1)$

Let a sample points (5, 1, 2, 8) and (0,1/4, 2/4, 3/4) then we find that sample points corresponds to the step function and j=0,1,2,3.

$$\tilde{f} = \sum_{j=0}^{n-1} y_j\, \varphi[x_j, x_{(j+1)})$$

$= 5\,\varphi[0,1/4) + 1\,\varphi[1/4,2/4) + 2\,\varphi[2/4,3/4) + 8\,\varphi[3/4,1)$

Discussion

The first step, $9\,\varphi[0, 1/2)$, has height 9 over the interval [0,1/2) starting at 0 and ending at ½. The second step, $1\,\varphi[1/2, 1)$, has height 1 over the interval [1/2,1) starting at ½ and ending at 1. The location [0,1/2) and [1/2,1) shows that the value of $\tilde{f}$ at ½ arises from $1\,\varphi[1/2, 1)$, which includes ½, but not from $9\,\varphi[0, 1/2)$, which excludes ½.

4.2(a) Let a sample points (x_0, y_0)= (0, 9) and (x_1, y_1)=(1/2, 1) then we find that sample points corresponds to the wavelet approximation and j=0,1.

$$\tilde{f} = (y_0+y_1)/2\,\varphi[0,1) + (y_0-y_1)/2\,\Psi[0,1)$$
$$= (9+1)/2\,\varphi[0,1) + (9-1)/2\,\Psi[0,1)$$
$$= 5\,\varphi[0,1) + 4\,\Psi[0,1)$$

Discussion: We have consider the two sample values y_0 and y_1 measure the value (amplitude, height) of the function $\tilde{f}$ at x_0 and x_1. In contrast the results from the basic approximation have the following conclusion from example, we have got the $\tilde{f} = 5\,\varphi[0,1) + 4\,\Psi[0,1)$

- The first step $5\,\varphi[0,1)$ means that the whole sample has an average value equals to 5
- The second step $4\,\Psi[0,1)$ means that from its first value to its second value, the sample changes as do 4 basic wavelets. Its under goes a jump of size 4 (-2) = - 8 effectively from 9 to 1.

Let consider a sample points (5, 1, 2, 8) and (0,1/4, 2/4, 3/4) then we find that sample points corresponds to the wavelet approximation and j=0,1,2,3.

$$\tilde{f} = 5\,\varphi[0,1/4) + 1\,\varphi[1/4,2/4) + 2\,\varphi[2/4,3/4) + 8\,\varphi[3/4,1)$$

The basic approximation applied to the first pair of steps gives,

$5\,\varphi[0,1/4) + 1\,\varphi[1/4,2/4) = 3\,\varphi[0,1/2) + 2\,\Psi[0,1/2)$

Similarly, after a shift by two sample points to the right, the basic approximation applied to the second pair gives

$2\varphi[2/4,3/4) + 8\,\varphi[3/4,1) = 5\,\varphi[1/2,1) - 3\,\Psi[1/2,1)$

Thus,

$$\tilde{f} = 5\,\varphi[0,1/4) + 1\,\varphi[1/4,2/4) + 2\,\varphi[2/4,3/4) + 8\,\varphi[3/4,1)$$
$$= 3\,\varphi[0,1/2) + 2\,\Psi[0,1/2) + 5\,\varphi[1/2,1) - 3\,\Psi[1/2,1)$$

Discussion: The coefficients 3, 5, 2, -3 have the following significance,

- $3\,\varphi[0,1/2)$ indicates that $\tilde{f}$ has an average values 3 over the first half of the interval from 0 to ½.
- $5\,\varphi[1/2,1)$ indicates that $\tilde{f}$ has an average values 5 over the second half of the interval from ½ to 1.
- $2\,\Psi[0,1/2)$ indicates that $\tilde{f}$ undergoes a jump of size 2 times that of $\Psi[0,1/2)$ which jumps down from 1 to -1, for a total of 2(-2)=-4 over the first half of the interval, indeed from 5 to 1.
- $(-3)\,\Psi[1/2,1)$ indicates that $\tilde{f}$ undergoes a jump of size (-3) times that of $\Psi[1/2,1)$ which jumps down from 1 to -1, for a total of (-3)(-2)=6 over the second half of the interval, indeed from 2 to 8.

The initial array $\vec{y}$ = (3,1,0,4,8,6,9,9)

This array contains the 2^3=8 values of the sample

$\vec{a}^{(3)} = \vec{y}$ = (3,1,0,4,8,6,9,9)

First sweep:

$\vec{a}^{(3-1)}$ = (2,2,7,9)
$\vec{c}^{(3-1)}$ = (1, -2, 1, 0)

This can be stored in the form

$\vec{y}^{(3-1)} = (\vec{a}^{(3-1)}, \vec{c}^{(3-1)})$ = (2,2,7,9,1,-2,1,0)

Second sweep:

$\vec{a}^{(3-1)}$ = (2,2,7,9)
$\vec{a}^{(3-2)}$ = (2,8)
$\vec{c}^{(3-2)}$= (0,-1)

This can be stored in the form

$\vec{y}^{(3-2)} = (\vec{a}^{(3-2)}; \vec{c}^{(3-2)}; \vec{c}^{(3-1)})$ = (2,8;0,-1;1,-2,1,0)

Third sweep:

$\vec{a}^{(3-2)}$ = (2,8)
$\vec{a}^{(3-3)}$ = (5)
$\vec{c}^{(3-3)}$ = (-3)

This can be stored in the form

$\vec{y}^{(3-3)} = (\vec{a}^{(3-3)}; \vec{c}^{(3-3)}; \vec{c}^{(3-2)}, \vec{c}^{(3-1)})$ = (5;-3;0,-1;1,-2,1,0)

The initial array $\vec{a}^{(3)} = \vec{y}$ represent the approximating function

$\tilde{f}$ *=3φ[0,1/8)+1φ[1/8,1/4)+0φ[1/4,3/8)+4φ[3/8,1/2)+*

8φ[1/2,5/8)+6φ[5/8,3/4)+9φ[3/4,7/8)+9φ[7/8,1)

$\tilde{f}$ *=5φ[0,1)+(-3)Ψ[0,1)+0Ψ[0,1/2)+(-1)Ψ[1/2,1)+1Ψ[0,1/4)+ (-2)Ψ[1/4,1/2)+1Ψ[1/2,3/4)+0Ψ[3/4,1)*

Discussion

- The term produced *5φ[0,1),* means that the sample has an average value equal to 5
- The penultimate term, *(-3)Ψ[0,1)*, indicates that the sample undergoes a jump 3 times the size of, and in the opposite direction from, the wavelet *Ψ[0,1)* (which jumps downward by 2 at the middle of the interval). Indeed, the sample jumps upward by 6 on average at the middle of the interval: The array $\vec{a}^{(3-2)}$ = (2,8) shows that the average jumps from 2 on the left hand half of the interval to8 on the right hand half of the interval.
- The two terms *0Ψ[0,1/2)+(-1)Ψ[1/2,1),* means that the sample does not exhibit any average jump at the first quarter of the interval, and exhibits an average jump of (-1)(-1)=1 at the third quarter.
- The four terms *1Ψ[0,1/4)+(-2)Ψ[1/4,1/2)+1Ψ[1/2,3/4)+0Ψ [3/4,1),* reveal that the sample oscillates, as do the fastest wavelets, with jumps of size -2, 4, -2,and 0.

Again considering the array of data from (4.3)

$\vec{y}^{(3)} = \vec{y}$ = (3,1,0,4,8,6,9,9)

Applying the In place Fast Haar Wavelet Approximation.

$\vec{y}^{(3-1)} = (a_0^{(3-1)}, c_0^{(3-1)}, a_1^{(3-1)}, c_1^{(3-1)}, a_2^{(3-1)}, c_2^{(3-1)}, a_3^{(3-1)}, c_3^{(3-1)})$

= (2, 1, 2, -2, 7, 1, 9, 0),

$\vec{y}^{(3-2)} = (a_0^{(3-2)}, c_0^{(3-1)}, c_0^{(3-2)}, c_1^{(3-1)}, a_1^{(3-2)}, c_2^{(3-1)}, c_1^{(3-2)}, c_3^{(3-1)})$

= (2, 1, 0, -2, 8, 1, -1, 0),

$\vec{y}^{(3-3)} = (a_0^{(3-3)}, c_0^{(3-1)}, c_0^{(3-2)}, c_1^{(3-1)}, c_0^{(3-3)}, c_2^{(3-1)}, c_1^{(3-2)}, c_3^{(3-1)})$

= (**5**, 1, 0, -2, -3, 1, -1, 0)

Next applying the In place Fast Inverse Haar Wavelet Approximation For the wavelet coefficients $\vec{y}^{(0)}$ = (4, 2, -1, -3)

The approximation gives

$$a_{2r}^{(l-1)} = a_r^{(l)} + c_r^{(l)}$$

$a_0^{(1)}$ = 4 + (-1) =3

$$a_{2r+1}^{(l-1)} = a_r^{(l)} - c_r^{(l)}$$

$a_1^{(1)}$ = 4 – (-1)=5

$\vec{y}^{(1)}$ = (3, 2, 5, -3)

$a_0^{(2)}$ = 3+2=5
$a_1^{(2)}$ = 3-2=1
$a_2^{(2)}$ = 5 + (-3)=2
$a_3^{(2)}$ = 5 – (-3)=8

which correctly reproduces the initial array $\vec{y}^{(2)}$ = (5, 1, 2, 8).

Lastly considering the data from sixteen numbers represent semi weekly measurements of temperature, in degrees Fahrenheit at a fixed common location of the period December and January (Ref-6). i.e

$\vec{y}^{(4)}$ = 32.0, 10.0, 20.0, 38.0, 37.0, 28.0, 38.0, 34.0, 18.0, 24.0, 18.0, 9.0, 23.0, 24.0, 28.0, 34.0

This data contains the 2^4=16 values of the samples,

By Applying In-place Fast Haar Wavelet Approximation

$\vec{y}^{(n)} = \vec{y}$ = (y_0, y_1, y_2, ... y_{2r}, y_{2r+1}, ... y_{2^n-2} , y_{2^n-1})

First Sweep:

$\vec{y}^{(n-1)}$ = [(y_0+y_1)/2, (y_0-y_1)/2,....., (y_{2r}+y_{2^r+1})/2 , (y_{2r} - y_{2^r+1})/2,...... (y_{2^n-2}+y_{2^n-1})/2 , (y_{2^n-2} - y_{2^n-1})/2]

= ($a_0^{(n-l)}$, $c_0^{(n-l)}$, $a_1^{(n-l)}$, $c_1^{(n-l)}$, $a_2^{(n-l)}$, $c_2^{(n-l)}$,…….., $a_r^{(n-l)}$, $c_r^{(n-l)}$,…….., $a_{2^{n-1}-1}^{(n-1)}$, $c_{2^{n-1}-1}^{(n-1)}$)

= (21.0, 11.0, 29.0, -9.0, 32.5, 4.5, 36.0, 2.0, 21.0, -3.0, 13.5, 4.5, 23.5, -0.5, 31.0, -3.0)

Second Sweep:

$\vec{y}^{(n-2)}$=[($a_0^{(n-l)}$+$a_1^{(n-l)}$)/2, $c_0^{(n-l)}$,($a_0^{(n-l)}$ − $a_1^{(n-l)}$)/2, $c_1^{(n-l)}$,……. ($a_{2r}^{(n-l)}$+$a_{2r+1}^{(n-l)}$)/2, $c_{2r}^{(n-l)}$,($a_{2r}^{(n-l)}$- $a_{2r+1}^{(n-l)}$)/2, $c_{2r+1}^{(n-l)}$,…….. ($a_{2^{n-1}-2}^{n-1}$ +$a_{2^{n-1}-1}^{n-1}$)/2, $c_{2^{n-1}-2}^{n-1}$, ($a_{2^{n-1}-2}^{n-1}$ - $a_{2^{n-1}-1}^{n-1}$)/2, $c_{2^{n-1}-1}^{n-1}$]

=($a_0^{(n-2)}$,$c_0^{(n-l)}$, $c_0^{(n-2)}$,$c_1^{(n-l)}$,$a_1^{(n-2)}$, $c_2^{(n-l)}$, $c_1^{(n-2)}$, $c_3^{(n-l)}$,$a_2^{(n-2)}$, $c_4^{(n-l)}$, $c_2^{(n-2)}$, $c_5^{(n-l)}$,…….., $c_{2^{n-2}-1}^{(n-2)}$, $c_{2^{n-1}-1}^{(n-1)}$)

= (**25.0**, 11.0, -4.0, -9.0, **34.25**, 4.5, -1.75, 2.0, **17.25**, -3.0, 3.75, 4.5, **27.25**, -0.5, -3.75, -3.0)

Similarly, Third Sweep and Fourth Sweep:

$\vec{y}^{(n-3)}$ = (**29.625**, 11.0, -4.0, -9.0, -4.625, 4.5, -1.75, 2.0, **22.25**, -3.0, 3.75, 4.5,-5, -0.5, -3.75, -3)

$\vec{y}^{(n-4)}$ = (**25.9375**, 11.0, -4.0, -9.0, -4.625, 4.5, -1.75, 2.0, 3.6875, -3.0, 3.75, 4.5, -5.0, -0.5, -3.75, -3.0)

The In-place Fast Haar Wavelet Approximation produces the result
25.9375;
3.6875;
-4.625, -5.0;
-4.0, -1.75, 3.75, -3.75;
11.0, -9.0, 4.5, 2.0, -3.0, 4.5, -0.5, -3.0.

Discussion

- The first coefficient, 25.9375, represents the average temperature for the whole two month period
- The second coefficient, 3.687, is the coefficient of the longest wavelet over the whole period, which means that the temperature changed by 3.687(-2)=-7.375, decrease of 7.75^{0}F,
- The next two coefficients, -4.625 and -5.0, represent similar changes of temperature over the first half (first two quarters) and over the second half(last two quarters) of the period, the coefficient -4.625 corresponds to a change of -4.625(-2)= 9.25, an increase of 9.25^{0}F from the first two weeks to the last two weeks in the Dec. The coefficient -5.0 corresponds to a change of -5.0(-2) =10, an increase of 10^{0}F from the first two weeks to the last two weeks in the Jan.
- Each of the next four coefficients, -4.0, -1.75, 3.75 and -3.725, represents change of temperature over two weeks, for instance, the coefficient -4.0 means that the temperature increased by -4.0(-2) =8^{0}F from the first week to the second week of Dec
- Finally, each of the last eight coefficients, 11.0, -9.0 4.5 2.0 -3.0 4.5 -0.5 -3.0 Represents change of temperature over one week, for instance, the coefficient 11.0 means that the temperature changed by 11.0(-2) =-22^{0}F during the first week of Dec; Indeed, the data show drop form 42^{0}F down 10^{0}F.

As a verification, the In place Fast Inverse Haar Wavelet Approximation reproduce the date exactly.

REFERENCES

1) Alfred, H., (1910), “Zur Theorie derorthogonalen Funktionen-Systeme”, Mathematische Annalen, Vol. 69, pp. 331-371.
2) Eugenio, H., and Guido W., (1996), “A First Course on Wavelets”, CRC Press, Boca Raton, FL, ISBN 0-8493-8274-2.
3) Gilbert, G. and Walter, (1994), “Wavelets and Other Orthonormal Systems With Applications”, CRC Press, Boca Raton, FL, ISBN 0-8493-7878-8.
4) Shiralashetti, S. C. and Kantli, M. H., (2013), “Numerical Solution of Typical Initial Value Problems using Haar Wavelet Transform Method”, Int. Jl. of Current Research, Vol. 5(5), pp-1168-1171.
5) Daubechies, I., (1992) “Ten Lectures On Wavelets”, SIAM, Philadelphia. PA, ISBN 0-89871-274-2.
6) Meyer, Y., (1990) “Ondelettes et Operateurs I: Ondelettes”, Hermann, paris, France, ISBN 2-7056-6125-0.
7) Heinz, W., Alfred, E., Louis, K. and William, R., (1997) “Inverse Problems in Geophysical Applications”, ISBN 0-89871-381-1.

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