Available online at http://www.journalcra.com

INTERNATIONAL JOURNAL OF CURRENT RESEARCH

International Journal of Current Research
Vol. 4, pp.135-139, May, 2010

ISSN: 0975-833X

RESEARCH ARTICLE

CORRELATED RANDOM VARIABLES IN MANPOWER PLANNING – A SHOCK MODEL APPROACH

Parthasarathy, S.

Department of Statistics, Annamalai University, Annamalai Nagar, Chidambaram- 608 002

ARTICLE INFO

Article History:
Received 12th April, 2010
Received in revised from
17th April, 2010
Accepted 26th April, 2010
Published online 11th May, 2010

Key words:
Correlated, Cumulative damage
Recruitment
Threshold
Shock model

ABSTRACT

Shock models it is usually assumed that the interarrival times between shocks are i.i.d random variables. In manpower model discussed here is assumed that there is loss of manpower to a random extent at every decision epoch at which revised policies regarding wages, incentives and targets are announced. When the cumulative loss of manpower on successive occasions crosses the threshold level, the breakdown of the organization occurs which in turn leads to recruitment. Here to discuss the model-1 is assumed that the interarrival times between successive decision epochs are not independent but correlated. However, the independence of the random amount of damages on successive decisions is not relaxed. In model-2 is assumed that the amount of depletion of manpower at every decision making epoch depends upon the interarrival time between the previous decision. For better understanding of the above model provided by the numerical illustration.

INTRODUCTION

The concept of shock model and cumulative damage processes has been used to determine the expected time to the breakdown of the organization under different assumptions, especially regarding the threshold distribution and the distribution of interarrival times between successive damages (the amount of loss of manpower). In doing so one of the basic assumptions of the model has been regarding the independence of the amount of damages caused on successive epochs and the interarrival times. Another assumption was that the amount of damage in a particular shock (decision) is independent of the time since the previous decision. Also it has been assumed that the interarrival times between the successive decisions epochs are i..i.d. This kind of independence of the variables involved can be relaxed and under the conditions of existence of correlation between the variables involved, the impact on the expected time to the breakdown of the organisation is discussed in the present study. Two different models under different conditions are discussed. One can referee shock model approach in Bartholomew (1971), Girnold and Marshall (1977), Bartholomew and Forbes (1979) and Parthasarathy (2002).

Model I

In the above described statement it is assumed that the random amount of manpower depletion on successive decision epochs are i.i.d. random variables. So also the interarrival times between successive decisions. But in the present model it is assumed that the interarrival times between successive decision epochs are not independent but correlated. However, the independence of the random amount of damages on successive decisions is not relaxed.

***Corresponding author:** *statsarathy@yahoo.com*

Under this assumption the expected time to the breakdown of the organisation and its variances are obtained.

Assumptions of the model

[1] On every occasion of decision making there is a random amount of manpower depletion.

[2] As and when the total amount of the depletion of manpower crosses a threshold level the organisation faces a situation of breakdown.

[3] The interarrival times between successive decision making epochs are not independent but correlated.

[4] The depletion of manpower on successive decision epochs are linear and cumulative.

Notations

X_i : a continuous random variable representing random amount of damage/depletion caused to the system due to the the ith occasion of exit of personnel, $i = 1, 2, \ldots k$.

$g(.)$: the probability density function of X.

$g_k(.)$: the k-fold convolution of g(.) i.e., p.d.f. of $\sum_{i=1}^{k} X_i$

Y : a continuous random variable representing the random threshold

$h(.)$: the p.d.f. of random threshold Y and H(.) is the corresponding c.d.f.

U_j : a continuous random variable representing the interarrival times between successive decisions.

$f(.)$: p.d.f. of Uj, j 1,2, …, k with the corresponding c.d.f. F(.)

$f_k(t)$: the k-fold convolution function of f(.).

$F_k(.)$: k convolution of F(.)

T : a continuous r.v denoting time to breakdown of the system.

ρ : Correlation between any X_i and X_j, i and j ϕ (k, υ).

RESULTS

In the case of the interarrival times U_i's being i.i.d., it has been shown that L (t) = P(T < t) = 1 – S(t), where T is the time to the breakdown of the organisation

S(t) = P [T > t]

$$= \sum_{k=0}^{\infty} V_k(t).P(\sum_{i=1}^{k} Ui < Y)$$

$$= \sum_{k=0}^{\infty} [V_k(t).[g^*(\theta)]^k \qquad (1)$$

L(t) = 1 – S(t)

$$= [1-g^*(\theta)] \sum_{k=1}^{\infty} [g^*(\theta)]^{k-1}.F_k(t)$$

$$= [1-g^*(\theta)] \sum_{k=1}^{\infty} [g^*(\theta)]^{k-1} F_k(t) \text{ on simplification} \qquad (2)$$

Now, it is assumed that the random variables U_i, i = 1, 2, … denoting the interarrival time between decisions are not i.i.d. but identically distributed and constantly correlated. For this purpose the results of Gurland (1955) are used. He has shown that the characteristic function ϕ($\lambda_1,\lambda_2, \ldots,\lambda_n$) of the joint distribution of any n random variables from a sequence $\{U_j\}/\{X_n\}$ of exchangeable random variables each following exponential distribution with p.d.f. f(x) = (1/α)exp(-u/α); α > 0 < U < ∝ such that the correlation coefficient ρ between any U_i and U_j ; i #j (independent of i and j) is given by

$$\phi(\lambda_1,\lambda_2, \ldots,\lambda_n) = \begin{bmatrix} 1-i\lambda_1\alpha & -i\lambda_1\alpha\rho & & -i\lambda_1\alpha\rho \\ -i\lambda_2\alpha\rho & 1-i\lambda_2\alpha & & -i\lambda_2\alpha\rho \\ & & & \\ -i\lambda_n\alpha\rho & & & 1-i\lambda_n\alpha \end{bmatrix}^{-1} \qquad (3)$$

[A sequence (U_j) ; j = 1,2, …… random variables is called a sequence of exchangeable random variables, if the joint distribution $F_k(u_1,u_2,\ldots,u_k)$ of any k-variables can be expressed as

$$F_k(u_1,u_2,\ldots,u_k) = \int_{\Omega} G_w(u_1)\ldots\ldots.G_w(u_k)dR(w), \quad k = 1,2,\ldots.$$

where $G_w(u)$ is a conditional distribution function of u for each w and a random variable in for a given u. Here Ω is the space of w].

Here the random variables U_i are taken as identically distributed, constantly correlated and exchangeable following exponential distribution.

Now, the distribution function of the partial sum S = $U_1 + U_2 + ..+U_k$ is defined as $F_k(U)$. Hence, the c.d.f. of $S_k = U_1 + U_2 + \ldots + U_k$ is

$F_k(u) = P(S_k \leq u)$

$$= (1-\rho)\sum_{i=0}^{\infty} \frac{(k\rho)^i}{[1-\rho+k\rho]^{i+1}} \frac{\varphi(k+i,u/b)}{(k+i-1)} \qquad (4)$$

where $\varphi(k,u) = \int_0^u e^{-\tau}\tau^{k-1}d\tau$ and $b = \alpha(1-\rho)$

Substituting (4) in (2) for $F_k(t)$ then,

$$L(t)=[1-g^*(\theta)\sum_{k=1}^{\infty}[g^*(\theta)]^{k-1}](1-\rho)\sum_{i=0}^{\infty} \frac{(k\rho)^i}{[1-\rho+k\rho]^{i+1}} \frac{\varphi(k+i,t/b)}{(k+i-1)}$$

$$=(1-\rho)[1-g^*(\theta)] \sum_{k=1}^{\infty}[g^*(\theta)]^{k-1} \sum_{i=0}^{\infty} \frac{(k\rho)^i}{[1-\rho+k\rho]^{i+1}} \frac{\varphi(k+i,t/b)}{(k+i-1)} \qquad (5)$$

Taking Laplace-Stieltjes transform of L(t),

$$L^*(s) = [1-g^*(\theta)] \frac{\sum_{k=1}^{\infty}[g^*(\theta)]^{k-1}\vartheta^k]}{\left[1+\frac{k\rho(1-\vartheta)}{(1-\rho)}\right]} \qquad (6)$$

where

$$\vartheta=\vartheta(s)=\frac{1}{1+bs} \; and \, \vartheta'(s)=-\frac{b}{(1+bs)^2}, \vartheta'(s)\big|_{s=0}=-b, b=\alpha(1-\rho).$$

Here

$$E(T) = -\frac{d}{ds}L^*(s)\Big|_{s=0} = \mu'_1$$

$$= \frac{\alpha}{1-g^*(\theta)} \text{ on simplification.} \qquad (7)$$

$$E(T^2) = \frac{d^2}{ds^2}L^*(s)\Big|_{s=0} = \mu'_2$$

$$= \left[\frac{2\alpha^2(1+\rho^2)}{(1-g^*(\theta))^2}\right] - \frac{2\alpha^2\rho^2}{(1-g^*(\theta))} \text{ on simplification.} \qquad (8)$$

$$= \frac{\alpha(\lambda+\theta)}{\theta} \quad \text{on simplification}$$

$$V(T) = \frac{\alpha^2+2\alpha^2\rho^2 g^*(\theta)}{(1-g^*(\theta))^2}. \qquad (9)$$

$$= \frac{\alpha^2(\lambda+\theta)}{\theta^2}\left[(\lambda+\theta)+2\rho^2\lambda\right]$$

It can be verified that ρ=0 then E(T) and V(T) for the case of U_i being i.i.d. random variables is obtained.

Model II

In the present model it is assumed that the amount of depletion of manpower at every decision making epoch depends upon the interarrival time between the previous decision and the present one. In otherwords, every time the policy announcements are made, there is a corresponding exit of personnel and the equivalent loss of man hour denoted as random variable X_i (damage at i^{th} epoch). The interarrival times between successive policy revisions are denoted as random variable Y_i. It is assumed that we have a sequence of pairs of random variables (X_n, Y_n) such that X_n and Y_n are correlated. For example if X_n denotes the amount of wastage in the n^{th}

Table 1.1

	θ=3			λ=2		
α	λ=2	λ=4	λ=6	θ=3	θ=5	θ=7
1	1.666667	2.333333	3	1.666667	1.4	1.285714
2	3.333333	4.666667	6	3.333333	2.8	2.571429
3	5	7	9	5	4.2	3.857143
4	6.666667	9.333333	12	6.666667	5.6	5.142857
5	8.333333	11.66667	15	8.333333	7	6.428571
6	10	14	18	10	8.4	7.714286
7	11.66667	16.33333	21	11.66667	9.8	9
8	13.33333	18.66667	24	13.33333	11.2	10.28571
9	15	21	27	15	12.6	11.57143
10	16.66667	23.33333	30	16.66667	14	12.85714

Figure 1.1

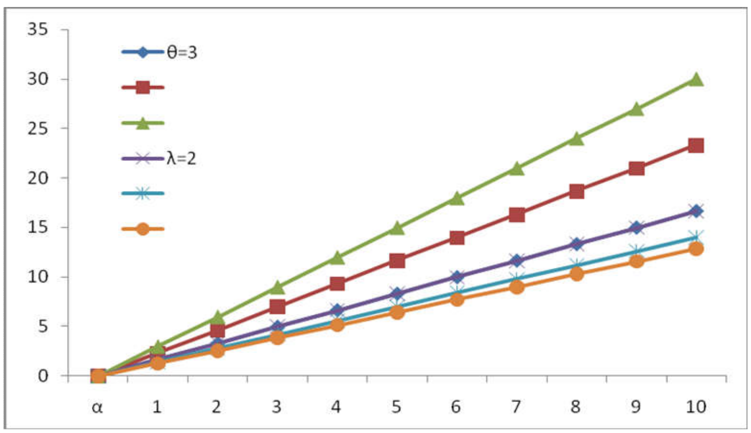


Table 1.2

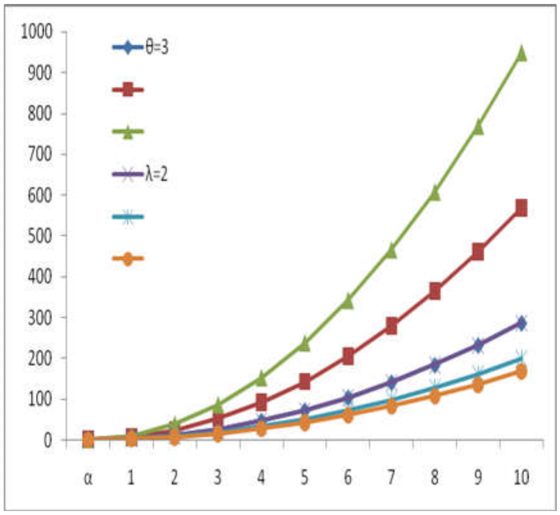
	θ=3, ρ=0.2			λ=2, ρ=0.2		
α	**λ=2**	**λ=4**	**λ=6**	**θ=3**	**θ=5**	**θ=7**
1	2.866667	5.693333	9.48	2.866667	2.0048	1.682449
2	11.46667	22.77333	37.92	11.46667	8.0192	6.729796
3	25.8	51.24	85.32	25.8	18.0432	15.14204
4	45.86667	91.09333	151.68	45.86667	32.0768	26.91918
5	71.66667	142.3333	237	71.66667	50.12	42.06122
6	103.2	204.96	341.28	103.2	72.1728	60.56816
7	140.4667	278.9733	464.52	140.4667	98.2352	82.44
8	183.4667	364.3733	606.72	183.4667	128.3072	107.6767
9	232.2	461.16	767.88	232.2	162.3888	136.2784
10	286.6667	569.3333	948	286.6667	200.48	168.2449

decision/policy revision, it is assumed that X_n depends upon Y_n where Y_n denotes the interarrival time between $(n\text{-}1)^{th}$ and n^{th} decisions.

For example if Yn is sufficiently larger then two possibilities could arise.

(i) Since the time interval Y_n is larger/longer the personnel attached to the organization would have put in more of Completed Length of Service (CLS), and hence the propensity to leave the job is likely to decrease, this results in a smaller number of persons leaving the job, and hence X_n is also less.

Figure 1.2



(ii) Since Y_n is longer, the expectations regarding the incentives would be naturally high on one hand and on the other the frustration during this interval would have increased in magnitude. Hence X_n is likely to be higher if the targets and incentives are not adequately awarded to the satisfaction of the employees.

Assumptions of the Model

1. The amount of damage X_n in the nth decision making epoch is correlated with Y_n which represents the time interval between the (n-1) th decision epoch and nth decision epoch.
2. The threshold level for the breakdown due to manpower depletion is denoted as Z which is a prespecified value and not a random variable.

Notations

X_n : The random variable denoting magnitude of the manpower loss due to the n^{th} decision.

Y_n : The random variable denoting the time interval between the $(n-1)^{th}$ and the n^{th} decision.

Z : The prespecified threshold level beyond which the system fails.

$F_{X,Y}$ (x, y) : joint marginal distribution function of X and Y.

$F_X(x)$, $F_Y(y)$: Marginal distribution function of X and Y respectively with corresponding density functions as $f_X(.)$ and $f_Y(.)$.

μ_x ,μ_y : Means of X and Mean of Y respectively.

μ_{x2} ,μ_{y2} : Second order moments of X and Y.

σ^2_x,σ^2_y : Variance of X ,Y respectively

S_z : The time to cross the threshold level (it has been represented as T in the previous chapters.

In this case the expected time to cross the threshold Z (Z is the breakdown point of the organization) can be obtained by using results of Shanthikumar and Sumitha (1983,1985). Here Z is the prespecified level of manpower depletion in manhours beyond which the system breaks down leading to recruitment. The expressions for $E(S_z)$ and its variance $V(S_z)$ are obtained using the results of Shanthikumar and Sumita (1983,1985).

It may be observed that S_Z denotes the time to breakdown of the system which hither to has been denoted as T previously.

It is given that

$$\int_0^{\infty} e^{-wz} E(S_z)dz = \frac{\eta_y}{w[1-\phi_x(w)]} \qquad (10)$$

where $$\phi_x(w) = \int_0^{\infty} e^{-wz} f_x(x)dx$$

Here, $E(S_z)$ can be obtained by using the above equation

$$\int e^{-wz} E(S_z)dz = \frac{\eta_y}{w[1-\phi_x(w)]}$$ by taking the Laplace inverse transform. (11)

Also

$$\int e^{-wz} E(S_z^2)dz = \frac{\eta_{\ y2}}{w[1-\phi_x(w)]} + \eta_y \sum_{n=0}^{\infty}(n+1)\phi_x(w)v(w) \qquad (12)$$

$$v(w) = 2\int_0^{\infty} e^{-wz} E(Y \mid X)Z \ dF_x(x)$$

Therfore $E(S_z^2)$ can be found by taking inverse Laplace transform.

If X~exp (λ), whose density function $f_x(x) = \lambda.e^{-\lambda}$

then $$\phi_x(w) = \int_0^{\infty} e^{-wx}.\lambda.e^{-\lambda x}.\,dx$$

$$= \lambda \int_0^{\infty} e^{-(w+\lambda)x}.\lambda\, dx$$

$$= \frac{\lambda}{w+\lambda} \quad \text{(or)} \quad \phi_x^*(w) = \frac{\lambda}{w+\lambda}$$

Substituting in (10),

we get $$\int_0^{\infty} e^{-wx}.E(S_z)dz = \frac{\mu_y}{w\left[1-\frac{\lambda}{\lambda+w}\right]} = \frac{\mu_y}{\frac{w^2}{w+y\lambda}}$$

$$\Rightarrow \frac{\lambda+w}{\mu.w^2} = \frac{\lambda+w}{\mu w^2}$$

Now to find $E(S_z)$

$$E(S_z) = L^{-1}\left[\frac{\lambda+w}{\mu w^2}\right]$$

$$= \frac{1}{\mu} L^{-1}\left[\frac{\lambda+w}{w^2}\right]$$

$$= \frac{1}{\mu}\left\{L^{-1}\left[\frac{\lambda}{w^2}\right]+L^{-1}\left[\frac{1}{w}\right]\right\}$$

$$= \frac{1}{\mu}\{1 + \lambda.Z\}$$

Therefore $E(S_z) = \dfrac{1+\lambda z}{w}$ So,

$$E(S_z) = \frac{1+\lambda z}{\alpha} \tag{13}$$

$$V(S_z) = \frac{1}{\alpha^2}\left[1+\alpha z(2-\beta/2)+\frac{\beta}{4}(e^{-2\lambda z}-1)\right] \tag{14}$$

DISCUSSION

Model I

It may be observed that the expected time to the breakdown of the organisation remains the same inspite of the fact that the interarrival times are correlated but the variance changes and it is a function of the correlation coefficient ρ. The variation in E(T) and V(T) are illustrated numerically in Table 1.1 and 1.2 respectively and the corresponding figures shown in Figures 1.1 and 1.2.

Model II

In the case of model-2 based on the Table 2.1 the corresponding figure indicate that as Z increases E(T) increases.

Table 2.1 λ=0.2,

Z	α=0.4	α=0.8	α=1.2
1	3	1.5	1
2	3.5	1.75	1.166667
3	4	2	1.333333
4	4.5	2.25	1.5
5	5	2.5	1.666667
6	5.5	2.75	1.833333
7	6	3	2
8	6.5	3.25	2.166667
9	7	3.5	2.333333
10	7.5	3.75	2.5

Table 2.2 α = 0.4: λ = 0.2

Z	β=0.5	β=0.7	β=0.9
1	11.00924	10.91293	10.81663
2	15.95745	15.84044	15.72342
3	21.18759	21.16263	21.13766
4	26.83831	27.07363	27.30895
5	33.11645	33.86303	34.60961
6	40.33061	41.96285	43.59509
7	48.94113	52.01758	55.09403
8	59.63479	64.9887	70.34262
9	73.43612	82.31057	91.18502
10	91.87355	106.123	120.3724

Figure 2.1

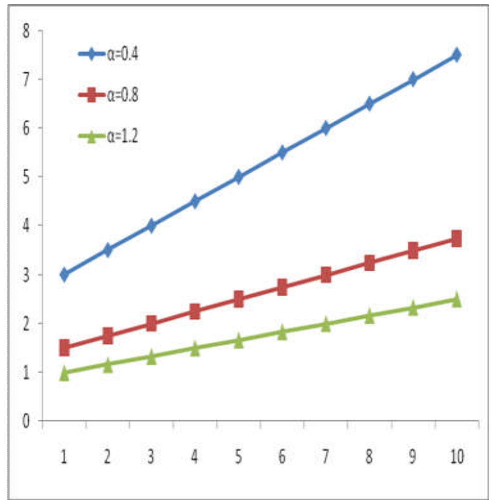
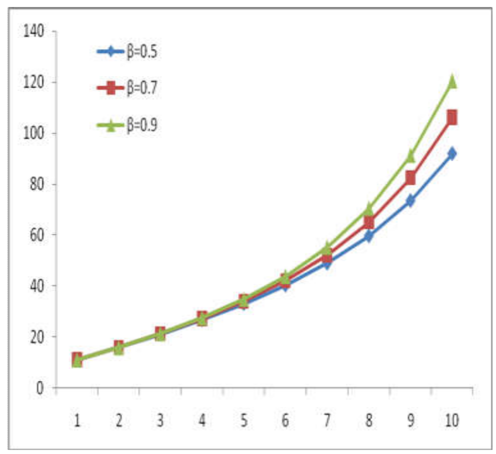


Figure 2.2



It is so, if λ also increases E(T) decreases when μ is fixed. The behavior of V(T) for the changes in Z for fixed ρ, μ as well as for different a values could be observed in Table 2.2 and the corresponding figures indicates V(T) increases. The E(T) decreases when α values changes keeping λ, μ are fixed.

Acknowledgement

The author thankfully acklowlwdge the help and guidance given by Dr. R. Sathyamoorthi, Retd. Professor and Head, Department of Statistics, Annamalai University and Dr. R. Ramanarayanan, Retd. Principal, Presidency College, Chennai.

REFERENCE

Barthlomew, D.J., 1971. The Statistical Approach to Manpower Planning. Statistician, 20: 3-26.

Bartholomew, D.J., and A.F Forbes, 1979. Statistical Techniques for Manpower Planning. John Wiley and Sons.

Esary, J.D., A.W. Marshall and F. Proschan, 1973. Shock models and wear processes. Ann. Probability, 1(4): 627-649.

Gurland, J. 1955. Distribution of the maximum of the arithmetic mean of correlated random variables. Ann. Mathematical Staticatics, 26: 294-300.

Shanthikumar, J.G and U. Sumitha, 1983. General Shock Models Associated with Correlated Renewal Sequences. Adv. Appl. Prob, 20: 600-614.

Shanthikumar, J.G and U. Sumitha, 1985. A Class of Correlated Cumulative Shock Models. Adv. Appl. Prob, 17:347-366.